JOURNAL OF ENGINEERING RESEARCH

V. 1 MARCH 1989

The Journal of Engineering Research is Published by The Engineering Research Center Tripoli, Jamahirya

CONTENTS

صياغة طريقة التكامل المتناهى المعدلة

د . مصطفى محمد الطويل

ملخص

تعتبر طريقة التكامل المتناهى احدى طرق التحليل العددى المصاغة حديثا ، فقد توصل براون وتراهير (مرجع 1) عام 1968م الى صياغة هذه الطريقة والتى تعتمد على اساس ان يتم تقريب اعلى تفاضل فى المسألة التفاضلية وكذلك التفاضلات الاقل ترتيبا ، كل على حدة ، بداول ثنائية الترتيب تكامل فيما بعد . وقد قام كاتب هذه الورقة بتطبيق هذه الطريقة الاصلية فى حل بعض مسائل ميكانيكا الانشاءات عند اعداد اطروحة الدكتوراة (مرجع 2) عام 1976م . وفى عام 1979 توصل الكاتب الى صياغة اولية لطريقة التكامل المتناهى المعدلة والتى تعتمد على التقريب عند اعلى تفاضل فقط بينما توجد التفاضلات الاقل عن طريق التكامل الفعلى للدالة التقريبية الاولى (مرجع 3) ، وقد كان التطبيق فيها مقتصرا على امثلة ذات المصفوفات ثلاثية الترتيب فقط .

فى هذه الورقة يتم عرض الخطوات المتبعة لصياغة طريقة التكامل المتناهى المعدلة صورتها العامة لتشمل مصفوفات بأى ترتيب ، كما تم تطبيق هذه الطريقة فى حل مسألة الترددات الطبيعية للاهتزازات الجانبية الحرة فى العوارض ، واخرى تتعلق بايجاد عزوم الانحناء والانحرافات فى العوارض المتناهية حرة الطرفين والموضوعة فوق اساسات مرنة تحت تأثير قوة مركزة عند احدى اطرافها ، لقد أعطت طريقة التكامل المتناهى المعدلة نتائج افضل اذا ما قورنت بتلك المتحصل عليها سواء عن طريق استعمال طريقة التكامل المتناهى الاصلية ام عن طريق استعمال طريقة الاختلاف المتناهى .

SUMMARY

A general formulation for the modified finite integral method is presented. The method is applied to two example problems governed by differential equations of the fourth order, and results obtained are compared with exact and some other numerical solutions.

•

EXPLANATIONS OF FIGURES AND TABLES

FORMULATION OF THE
MODIFIED FINITE INTEGRAL METHOD

By; Dr. Mustafa M. Tawil
Professor, Civil Engg. Dept.
University of El-Fateh

Introduction:

The finite integral method was first derived by Brown and Trahair (1) in 1968. They considered approximating the highest and subsequent lower derivatives of a given ordinary differential equation as being functions of second order variation at each stage; while the author in a previous paper (3) considered the highest derivative only to be approximated by a second degree variation in the independent variable, and the other lower derivatives as being exact integrations of that approximate variation.

This modified finite integral method was formulated earlier in 1979 by the author and applied to some problems using matrices of the third order only (3).

In this paper a general formulation of the modified finite integral method is presented and some example problems are solved using matrices of the order higher than three. The numerical results obtained by this method are compared with those of other numerical methods. This numerical comparative study reveals excellent accuracy of the proposed method.

Matrix Formulation:

If we let f(x) to be the approximating function to the highest derivative of a given ordinary differential equation; and I_i^x be the i^{th} integration performed on this function between the limits of 0 and x.

$$I_i^x = \underbrace{\int_0^x\int_0^x\int_0^x \cdots\cdots \int_0^x}_{i \text{ times}} f(x)\ \underbrace{dx\ dx\ dx \ldots\ldots dx}_{i \text{ times}}$$

and if the range of integration is divided into even number of equal segments, n, of length h; then in matrix form:

$$\{ I_i^x \} \simeq \left(\frac{h}{12}\right)^i [N_i] \{f\} \quad \ldots\ldots\ldots \quad (1)$$

Where

$$\{ I_i^x \} = \{ I_i^0 \quad I_i^h \quad I_i^{2h} \ldots\ldots \quad I_i^{nh} \}^T$$

$[N_i]$ = the i^{th} modified finite integral transformation matrix

$$\{ f \} = \{ f(0)\ f(h)\ f(2h)\ \ldots\ldots\ f(nh) \}^T$$

$$i = 1, 2, 3, \ldots\ldots\ldots$$

When the range of integration is divided into two equal segments, then using the modified finite integral method, matrices needed for solving upto the fourth order ordinary differential equation are given as (3):

$$[N_1] = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 8 & -1 \\ 4 & 16 & 4 \end{bmatrix} \quad \ldots\ldots\ldots \quad (2$$

$$[N_2] = \begin{bmatrix} 0 & 0 & 0 \\ 42 & 36 & -6 \\ 96 & 192 & 0 \end{bmatrix} \quad \text{.......(2b)}$$

$$[N_3] = \begin{bmatrix} 0 & 0 & 0 \\ 194.4 & 115.2 & -21.6 \\ 1036.8 & 1382.4 & -115.2 \end{bmatrix} \quad \text{....... (2c)}$$

$$[N_4] = \begin{bmatrix} 0 & 0 & 0 \\ 633.6 & 288 & -57.6 \\ 7372.8 & 7372.8 & -921.6 \end{bmatrix} \quad \text{........(2d)}$$

$$\{I_i^x\} = \{ I_i^0 \quad I_i^h \quad I_i^{2h} \}^T \quad \text{........(2e)}$$

$$\{f\} = \{ f(0) \quad f(h) \quad f(2h) \} \quad \text{........(2f)}$$

When more accuracy is required, the even number of equal segments is increased. The integrals over these segments may be derived using the following recurrence relations:

$$I_i^{(2m-1)h} = \sum_{k=0}^{i-1} \frac{h^k}{k!} I_{i-k}^{2(m-1)h} + I_i^h(m) \quad \text{....... (3a)}$$

$$I_i^{(2m)h} = \sum_{k=0}^{i-1} \frac{(2h)^k}{k!} I_{i-k}^{2(m-1)h} + I_i^{2h}(m) \quad \text{....... (3b)}$$

Where

m = 2, 3, 4,..........., n/2

n = number of equal segments (always even)

$0!$ = 1

$I_i^h(m)$ = the i^{th} integration of f(x) between the limits 0 and h of the two segments numbered m.

$I_i^{2h}(m)$ = the i^{th} integration of f(x) between the limits 0 and 2h of the two segments numbered m.

The last two are given as:

$$I_i^h(m) \simeq \left(\frac{h}{12}\right)^i \left[N_i(2,1)\, f\{(2m-2)h\} + N_i(2,2)\, f\{(2m-1)h\} + N_i(2,3)\, f(2mh) \right] \quad \ldots\ldots\ldots(4a)$$

$$I_i^{2h}(m) \simeq \left(\frac{h}{12}\right)^i \left[N_i(3,1)\, f\{(2m-2)h\} + N_i(3,2)\, f\{(2m-1)h\} + N_i(3,3)\, f(2mh) \right] \quad \ldots\ldots\ldots(4b)$$

To show how relations (3a, 3b, 4a, and 4b) are used, let us find I_2^{3h} and I_2^{4h}:

For this, i = 2; m = 2

From (3a):

$$I_2^{3h} = \left(\sum_{k=0}^{1} \frac{h^k}{k!} I_{2-k}^{2h} \right) + I_2^h(2)$$

$$= \left(I_2^{2h} + h\, I_1^{2h} \right) + I_2^h(2)$$

Making use of equations (1), (2a) and (2b):

$$I_2^{2h} \simeq \left(\frac{h}{12}\right)^2 \left[96f(0) + 192f(h) \right]$$

$$I_1^{2h} \simeq \left(\frac{h}{12}\right) \left[4f(0) + 16f(h) + 4f(2h) \right]$$

$$I_2^h(2) \simeq \left(\frac{h}{12}\right)^2 \left[42f(2h) + 36f(3h) - 6f(4h) \right]$$

then:

$$I_2^{3h} \simeq \left(\frac{h}{12}\right)^2 \left[96f(0) + 192f(h) + 48f(0) + 192f(h) + 48f(2h) + 42f(2h) + 36f(3h) - 6f(4h) \right]$$

$$\simeq \left(\frac{h}{12}\right)^2 \left[144f(0) + 384f(h) + 90f(2h) + 36f(3h) - 6f(4h) \right]$$

$$I_2^{3h} \simeq \frac{h^2}{24} [24f(0) + 64f(h) + 15f(2h) +$$

$$6f(3h) - f(4h)]$$

From 3(b): $$I_2^{4h} = \left(\sum_{k=0}^{1} \frac{(2h)^k}{k!} I_{2-k}^{2h} \right) + I_2^{2h} (2)$$

$$= (I_2^{2h} + 2h \, I_1^{2h}) + I_2^{2h} (2).$$

Making use of equations (1) and (2b):

$$I_2^{2h} (2) \simeq (\frac{h}{12})^2 [96f(2h) + 192f(3h)]$$

then: $$I_2^{4h} \simeq (\frac{h}{12})^2 [96f(0) + 192f(h) + 96f(0) + 384f(h)$$

$$+ 96f(2h) + 96f(2h) + 192f(3h)]$$

$$\simeq (\frac{h}{12})^2 [192f(0) + 576f(h) + 192f(2h) +192f(3h)]$$

or $$I_2^{4h} \simeq \frac{h^2}{72} [32f(0) + 96f(h) + 32f(2h) + 32f(3h)]$$

Fortran computer subroutines are given in Appendix IV which can be used to find the elements needed for 1st, 2nd 3rd and 4th integrals to cover any even numbers of segments. Higher order integrations may be handled in a similar way.

In view of Eqs. (2a, b, c, and d), one may write $[N_1]$, $[N_2]$, $[N_3]$ and $[N_4]$ as:

$$[N_1] = [N1]$$
$$[N_2] = 6[N2]$$
$$[N_3] = 7.2[N3]$$
$$[N_4] = 57.6[N4]$$

Where

$$[N1] = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 8 & -1 \\ 4 & 16 & 4 \end{bmatrix}$$

$$[N2] = \begin{bmatrix} 0 & 0 & 0 \\ 7 & 6 & -1 \\ 16 & 32 & 0 \end{bmatrix}$$

$$[N3] = \begin{bmatrix} 0 & 0 & 0 \\ 27 & 16 & -3 \\ 144 & 192 & -16 \end{bmatrix}$$

$$[N4] = \begin{bmatrix} 0 & 0 & 0 \\ 11 & 5 & -1 \\ 128 & 128 & -16 \end{bmatrix}$$

then, Eq. (1) can be represented as:

$$\{I_1^x\} \simeq \frac{h}{12} [N1] \{f\} \quad .... \qquad (5a)$$

$$\{I_2^x\} \simeq \frac{h^2}{24} [N2] \{f\} \quad ...... \qquad (5b)$$

$$\{I_3^x\} \simeq \frac{h^3}{240} [N3] \{f\} .... \qquad (5c)$$

$$\{I_4^x\} \simeq \frac{h^4}{360} [N4] \{f\} \quad .... \qquad (5d)$$

The above transformation is performed in order to remove common factors from the elements of $[N_2]$, $[N_3]$, and $[N_4]$ matrices.

The elements of the matrices [N1], ..., [N4] are given in Appendix I for ten equal segments. Elements needed for higher order integrals may be derived in the same manner.

Applications of the Method

a) Natural Frequencies of a Transversely Vibrating Beam:

The governing equation of motion for the beam shown in Figure 1 is:

$$\frac{\partial^2 y}{\partial t^2} + b^2 \frac{\partial^4 y}{\partial x^4} = 0 \; .... \qquad (6)$$

Where

y = rectangular coordinate representing the transverse displacement

x = rectangular coordinate representing the longitudinal distance from the origin

t = time variable

b^2 = EIg/w

EI = beam flexural rigidity

g = acceleration due to gravity

w = weight of the beam per unit length

Equation (6) can be separated into two ordinary differential equations by letting $y = XT$ to get:

$$\frac{d^4 X}{dx^4} - \frac{p^2}{b^2} X = 0 \; .... \qquad (7)$$

$$\frac{d^2 T}{dt^2} + p^2 T = 0 \; .... \qquad (8)$$

where

X = function of x only

T = function of t only

p = the natural frequency of the vibrating beam

Equation (7) is to be solved here using the modified finite integral method to obtain the natural frequencies of the beam.

Let $$\frac{d^4X}{dx^4} = f(x) \quad .... \qquad (9a)$$

then

$$\frac{d^3X}{dx^3} = I_1^x + A_1 \quad .... \qquad (9b)$$

$$\frac{d^2X}{dx^2} = I_2^x + A_1x + A_2 \quad .... \qquad (9c)$$

$$\frac{dX}{dx} = I_3^x + A_1x^2/2 + A_2x + A_3 \quad ... \qquad (9d)$$

$$X = I_4^x + A_1x^3/6 + A_2x^2/2 + A_3x + A_4 \quad .. \qquad (9e)$$

Where

A_1, A_2, A_3, and A_4 are constants of integration.

Using the boundary conditions:

at $x = 0$ and L, $X = EI\frac{d^2X}{dx^2} = 0$

one has:

$$A_1 = - I_2^L/L$$

$$A_2 = 0$$

$$A_3 = -\frac{1}{L}\,(I_4^L - L^2\, I_2^L/6)$$

$$A_4 = 0$$

Where

L = length of the span of beam

In matrix form, Eq. (7) becomes:

$$([M1] - \gamma[I])\,\{f\} = \{0\} \quad .... \qquad (10)$$

where

$$[M1] = -\frac{1}{360n^4}([N4] - [x/L][N4]_L)$$

$$+ \frac{1}{144n^2}([x/L]^3 - [x/L])\ [N2]_L$$

$\gamma = b^2/p^2L^4$

n = the number of equal segments into which the span is divided

$[I]$ = the identity matrix

$[x/L]$ = a diagonal matrix of the ratios x/L

$[x/L]^3 = [x/L][x/L][x/L]$

$[Ni]_L$ = a matrix with all the rows being identical to the row corresponding to $x = L$ in $[Ni]$.

Equation (10) is solved for the eigenvalues, γ_k, from which, the natural frequencies of the beam, p_k, can be calculated as:

$$p_k = \frac{1}{\sqrt{\gamma_k}} \cdot \frac{b}{L^2}, \qquad k = 1,2,3,$$

or in another form

$$p_k = \alpha_k \sqrt{\frac{EIg}{w}}\,/L^2 \quad \ldots \qquad (11)$$

where

$$\alpha_k = 1/\sqrt{\gamma_k}$$

Table 1 gives the values of α_1, α_2, and α_3 resulted from the solution of Eq. (10) using the modified finite integral method with 6, 8, and 10 equal segments. Results are compared with those using the original finite integral and finite difference methods (2).

Table 1.

Mode Number (1)	α (Exact) (2)	n (3)	Modified Finite Integral Method		Original Finite Integral Method (a)		Finite Difference Method (a)	
			α (4)	% error (5)	α (6)	% error (7)	α (8)	% error (9)
1	π^2	6	9.876	0.065	9.901	0.318	9.646	-2.266
		8	9.872	0.024	9.880	0.105	9.743	-1.283
		10	9.870	0.004	9.874	0.045	9.789	-0.817
2	$4\pi^2$	6	39.826	0.880	41.252	4.493	36.000	-8.811
		8	39.595	0.295	40.073	1.506	37.490	-5.037
		10	39.528	0.126	39.731	0.640	38.200	-3.238
3	$9\pi^2$	6	87.456	-1.543	108.000	21.585	72.000	-18.943
		8	90.023	1.347	95.070	7.030	79.017	-11.043
		10	89.355	0.595	91.503	3.013	82.443	-7.186

[a] Reference (2)

b) Finite Beam on Elastic Foundation:

A free-free beam on an elastic foundation with modulus k carrying a concentrated vertical load P at the left end (Figure 2) is solved here using the modified finite integral method.

The governing equation for this kind of problem is:

$$\frac{d^4y}{dx^4} + 4\alpha^4 y = 0 \qquad (12)$$

Where

$\alpha^4 = k/4EI$

y = rectangular coordinate representing the vertical deflection of the beam

The boundary conditions are given as:

$$\left.\begin{aligned} \text{at } x = 0;\quad & \frac{d^3y}{dx^3} = P/EI, \quad \frac{d^2y}{dx^2} = 0 \\ \text{at } x = L;\quad & \frac{d^3y}{dx^3} = 0, \quad \frac{d^2y}{dx^2} = 0 \end{aligned}\right\} \qquad (13)$$

The exact solution for Eq. (12) with the boundary conditions Eq. (13) is given as (4):

$$y = \frac{2P\alpha}{k(\sinh^2\alpha L - \sin^2\alpha L)} \left[\sinh\alpha L \cos\alpha L(x/L)\cosh\alpha L(1-x/L) - \sin\alpha L\cosh\alpha L(x/L)\cos\alpha L(1-x/L)\right] \quad \ldots\ldots \quad (14)$$

To solve Eq. (12) by the modified finite integral method, one proceeds in the same way as was done in example (a):

$$\text{Let} \quad \frac{d^4y}{dx^4} = f(x) \quad \ldots\ldots \qquad (15a)$$

then

$$\frac{d^3y}{dx^3} = I_1^x + A_1 \quad \ldots\ldots \tag{15b}$$

$$\frac{d^2y}{dx^2} = I_2^x + A_1x + A_2 \quad \ldots\ldots \tag{15c}$$

$$\frac{dy}{dx} = I_3^x + A_1x^2/2 + A_2x + A_3 \quad \ldots\ldots \tag{15d}$$

$$y = I_4^x + A_1x^3/6 + A_2x^2/2 + A_3x + A_4 \quad . \tag{15e}$$

Applying the boundary conditions, Eq. (13), in Eqs. (15b) and (15c) one has:

$$\begin{aligned} A_1 &= P/EI \\ A_2 &= 0 \\ A_1 &= -I_2^L/L \\ A_1 &= -I_1^L \end{aligned} \tag{16}$$

To determine A_3 and A_4, one lets $\frac{dy}{dx} = \theta_L$ and $y = y_L$ at $x = L$; then:

$$y_L = I_4^L + A_1L^3/6 + A_3L + A_4$$

$$\theta_L = I_3^L + A_1L^2/2 + A_3$$

where

θ_L = the slope at $(x = L)$

y_L = the deflection at $(x = L)$

Solving for A_3 and A_4 one has:

$$\left.\begin{aligned} A_3 &= -I_3^L - PL^2/2EI + \theta_L \\ A_4 &= -I_4^L + LI_3^L + PL^3/3EI - L\theta_L + y_L \end{aligned}\right\} \tag{17}$$

Taking A_1 as the first relation in Eq. (16), along with A_2, A_3, and A_4, the governing Eq. (12) can be written in matrix form as:

$$[M2]\{f\} \simeq -4\alpha^4 \left(\frac{PL^3}{6EI} \{ 2-3(x/L) + (x/L)^3 \} - \theta_L L \{ 1-x/L \} + y_L \{ 1 \} \right) \quad .... \quad (18)$$

Where

$$[M2] = [I] + 4(\alpha L)^4 \left[\frac{1}{360n^4} ([N4] - [N4]_L) + \frac{1}{240n^3} [(1-x/L)] [N3]_L \right]$$

Solution of Eq. (18) for {f} yields

$$\{f\} \simeq 4\alpha^4 \left(-\frac{P}{kL} \{f_1\} + \theta_L L \{F_2\} - y_L \{f_3\} \right) \quad .... \quad (19)$$

Where

$$\{F_1\} = \frac{2}{3} (\alpha L)^4 [M2]^{-1} \{2-3(x/L) + (x/L)^3\}$$

$$\{F_2\} = [M2]^{-1} \{1 - x/L\}$$

$$\{F_3\} = [M2]^{-1} \{1\}$$

For a given value of αL, the vector {f} is determined from Eq. (19) in terms of the two unknowns ($\theta_L L$) and (y_L). To find these unknowns, the vector {f} is substituted in the right hand sides of the last two relations of Eq. (16) with the left hand sides being equal to P/EI; that is:

$$P/EI = -I_2^L/L$$

$$P/EI = -I_1^L$$

When solving these two equations simultaneously, values of $\theta_L L$ and y_L are determined, and the last two unknown integration constants can now be determined by Eq. (17).

Table 2. gives the results of (kL^3 x bending moment/P) and (kL x deflection/P) for the left half of the beam with $\alpha L = 10$, with the span being divided into ten equal segments. Numerical solutions from the modified and the original finite integral methods are tabulated; and comparisons with the exact solution are given.

Conclusions:

A close examination of the results of the two example problems given in Tables 1 and 2 shows the following:

1. The overall results given by the modified finite integral method are much better than those given by the original finite integral method.

2. Both finite integral methods give better results than the finite difference method in the first example.

3. It seems that the modified finite integral method is an accurate numerical tool in solving problems, governed by ordinary differential equations or by partial differential equations that have the property of being reduced to a set of ordinary differential equations through a separation-of-variables technique.

Table 2.

x/L	kL^3 x Bending Moment/P					kL x Deflection/P				
	Exact	Modified Finite Integral Method	% error	Original Finite Integral Method	% error	Exact	Modified Finite Integral Method	% error	Original Finite Integral Method	% error
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
0	0	0	0	0	0	20	20.07	0.35	20.00	0
0.1	1238	1236	-0.162	1377	11.228	3.975	4.039	1.610	4.262	7.220
0.2	492.2	493.8	0.325	393.4	-20.073	-1.126	-1.098	-2.487	-1.639	455.560
0.3	28.10	32.39	15.267	-29.02	-203.3	-0.9858	-0.9779	-0.801	-1.027	4.179
0.4	-55.45	-59.27	6.889	-64.5	16.321	-0.2394	-0.2151	-10.150	-0.05912	-75.305
0.5	-25.84	-25.26	-2.245	-17.82	-31.037	0.03823	0.04758	24.457	0.09843	157.5

Note: $\alpha L = 10$ and $\eta = 10$

Matrix [N3] =

0	0	0	0	0	0	0	0	0	0	0
27	16	-3	0	0	0	0	0	0	0	0
144	192	-16	0	0	0	0	0	0	0	0
344	672	51	16	-3	0	0	0	0	0	0
624	1472	288	192	-16	0	0	0	0	0	0
984	2592	688	672	51	16	-3	0	0	0	0
1424	4032	1248	1472	288	192	-16	0	0	0	0
1944	5792	1968	2592	688	672	51	16	-3	0	0
2544	7872	2848	4032	1248	1472	288	192	-16	0	0
3224	10272	3888	5792	1968	2592	688	672	51	16	-3
3984	12992	5088	7872	2848	4032	1248	1472	288	192	-16

Matrix [N4] =

0	0	0	0	0	0	0	0	0	0	0
11	5	-1	0	0	0	0	0	0	0	0
128	128	-16	0	0	0	0	0	0	0	0
484	736	-9	5	-1	0	0	0	0	0	0
1200	2304	224	128	-16	0	0	0	0	0	0
2396	5312	936	736	-9	5	-1	0	0	0	0
4192	10240	2368	2304	224	128	-16	0	0	0	0
6708	17568	4760	5312	936	736	-9	5	-1	0	0
10064	27776	8352	10240	2368	2304	224	128	-16	0	0
14380	41344	13384	17568	4760	5312	936	736	-9	5	-1
19776	58752	20096	27776	8352	10240	2368	2304	224	128	-16

Appendix I Matrices [Ni]:

Matrix [N1] =

0	0	0	0	0	0	0	0	0	0	0
5	8	-1	0	0	0	0	0	0	0	0
4	16	4	0	0	0	0	0	0	0	0
4	16	9	8	-1	0	0	0	0	0	0
4	16	8	16	4	0	0	0	0	0	0
4	16	8	16	9	8	-1	0	0	0	0
4	16	8	16	8	16	4	0	0	0	0
4	16	8	16	8	16	9	8	-1	0	0
4	16	8	16	8	16	8	16	4	0	0
4	16	8	16	8	16	8	16	9	8	-1
4	16	8	16	8	16	8	16	8	16	4

Matrix [N2] =

0	0	0	0	0	0	0	0	0	0	0
7	6	-1	0	0	0	0	0	0	0	0
16	32	0	0	0	0	0	0	0	0	0
24	64	15	6	-1	0	0	0	0	0	0
32	96	32	32	0	0	0	0	0	0	0
40	128	48	64	15	6	-1	0	0	0	0
48	160	64	96	32	32	0	0	0	0	0
56	192	80	128	48	64	15	6	-1	0	0
64	224	96	160	64	96	32	32	0	0	0
72	256	112	192	80	128	48	64	15	6	-1
80	288	128	224	96	160	64	96	32	32	0

Appendix II. References

1. Brown, P. T. and Trahair, N. S., "Finite Integral Solution of Differential Equations," Civil Engineering Transactions, Institution of Engineers, Australia, Vol. CE10, No. 2, Oct. 1968, pp. 193-196.
2. Tawil, M. M., "The Finite Integral Method Applied to Some Problems of Structural Mechanics," thesis presented to the University of Colorado, at Boulder, Colorado, in 1976, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
3. Tawil, M. M., "Finite Integral Method (A Modified Approach)," Al-Fateh University Bulletin of the Faculty of Engineering, Tripoli, S.P.L.A.J., Vol. 3, Third Issue, 1979, pp. 87-93.
4. Timoshenko, S. P., "Strength of Materials," Part II, 2nd Ed., D. Van Nostrand Co., Inc., New York, 1941, pp. 1-25.

Appendix III. Notation

The following symbols are used in this paper:

A_1, A_2, A_3, A_4 = constants of integration;

b = factor for a vibrating beam;

EI = beam flexural rigidity;

$\{F_1\}, \{F_2\}, \{F_3\}$ = vectors;

$f(x)$ = function of x;

$\{f\}$ = vector of f(x);

g = acceleration due to gravity;

h = length of a segment;

$[I]$ = the identity matrix;

I_i^x = i^{th} integration performed on f(x) between the limits of 0 and x;

i = counter;

k = counter; modular of an elastic foundation;

L = span length;

[M1], [M2] = matrices;

m = counter;

$[N_i]$, [Ni] = matrices for the i^{th} integration of f(x);

$[Ni]_L$ = matrix with all its rows being identical to the row corresponding to x = L in [Ni];

n = number of equal segments into which the span L is divided;

P = concentrated load;

p, p_k = natural frequency;

T = function of t;

t = time variable'

w = weight of a beam per unit length;

X = function of x;

x = coordinate;

$\lceil x/L \rfloor$ = diagonal matrix with values of x/L;

y = coordinate; displacement;

y_L = deflection at x = L;

α = factor for a beam on elastic foundation;

α_k = factor;

γ_k = eigenvalue;

θ_L = slope at x = L.

FIG. 1

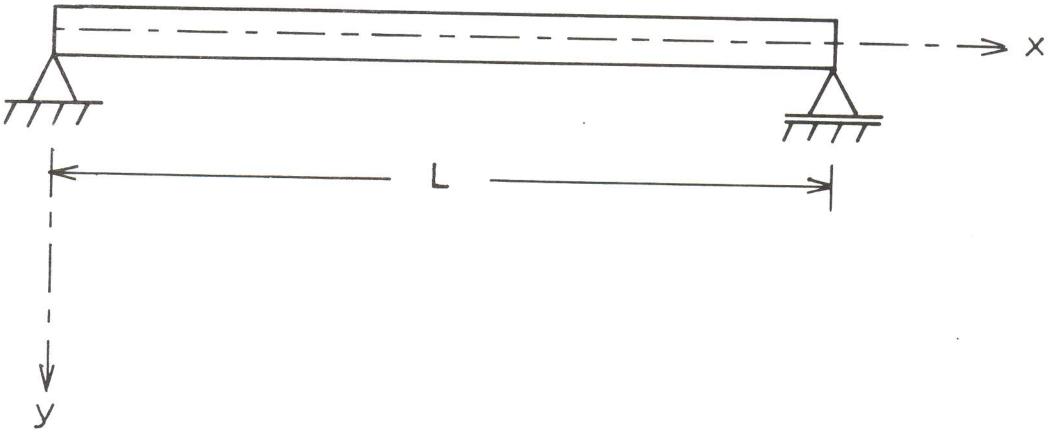
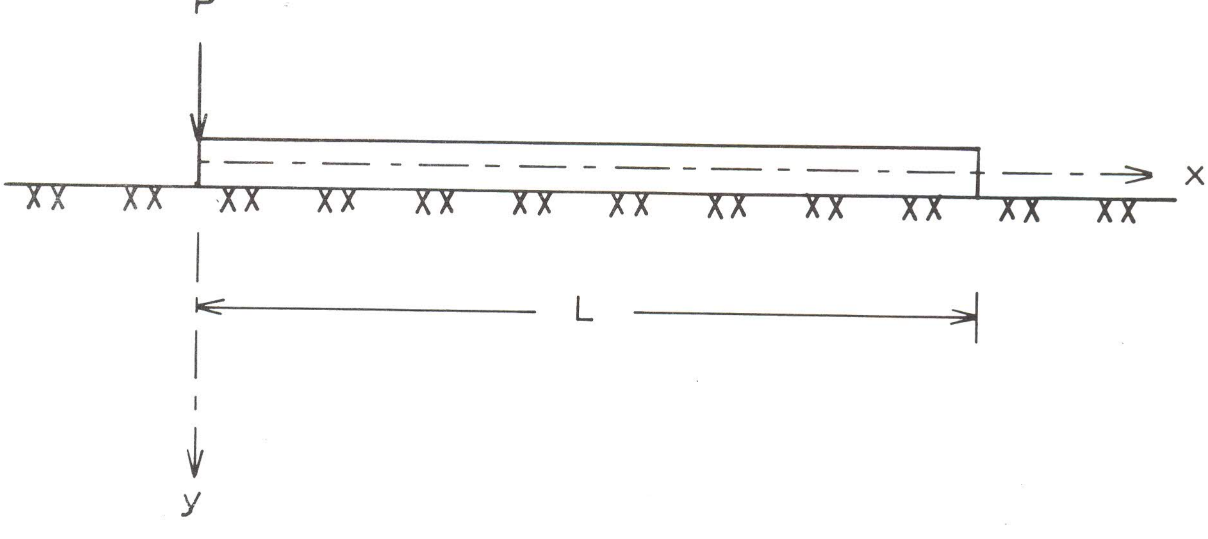


FIG.2



Appendix IV. Fortran Computer Subroutines

The following subroutines may be used to find the elements of matrices for integration over any even number of segments using finite integral method:

FSTGAL: A subroutine to find the elements of the first integral matrix [A], where:

$$\{I_1^x\} = [A]\{f\}$$

SNDGAL: A subroutine to find the elements of the second integral matrix [B], where:

$$\{I_2^x\} = [B]\{f\}$$

TRDGAL: A subroutine to find the elements of the third integral matrix [C], where:

$$\{I_3^x\} = [C]\{f\}$$

FRTGAL: A subroutine to find the elements of the fourth integral matrix [D], where:

$$\{I_4^x\} = [D]\{f\}$$

It should be noted that each integral matrix will use the elements of all integral matrices lower than the order required. That means if matrix [C] is needed, then matrices [A] and [B] are required in finding the elements of [C], see equations (3a and 3b).

The symbols used in these subroutines are given below:

N is the number of segments in the range of integration, always even.

XL is the length over which the integration is required.

A2 is the matrix [N1] for two segments.

A is thematrix [A]

NA is the dimensions of matrix [A] in the main program.

B2 is the matrix [N2] for two segments.

B is the matrix [B].

NB is the dimensions of matrix [B] in the main program.

C2 is the matrix [N3] for two segments.

C is the matrix [C].

NC is the dimensions of matrix [C] in the main program.

D2 is the matrix [N4] for two segments.

D is the matrix [D].

ND is the dimensions of matrix [D] in the main program.

```
      SUBROUTINE FSTGAL(N,XL,A2,A,NA)
      DIMENSION A2(3,3),A(NA,NA)
      M= N+1
      H= XL/N
      DO 1 I=1,M
      DO 1 J=1,M
    1 A(I,J)= 0.
      F1= H/12.
      DO 2 I=2,3
      DO 2 J=1,3
    2 A(I,J)= F1*A2(I,J)
      IF(N.EQ.2) GO TO 10
      N2= N/2
      DO 3 I=2,N2
      I1= 2*I
      I2= I1-1
      I3= I2-1
      I4= I1+1
      DO 4 J=1,I3
      A(I1,J)= A(I2,J)
    4 A(I4,J)= A(I1,J)
      A(I1,I2)= A(I2,I2)+A(2,1)
      A(I4,I2)= A(I2,I2)+A(3,1)
      A(I1,I1)= A(2,2)
      A(I1,I4)= A(2,3)
      A(I4,I1)= A(3,2)
    3 A(I4,I4)= A(3,3)
   10 RETURN
      END
```

```
      SUBROUTINE SNDGAL(N,XL,B2,B,NB,A)
      DIMENSION B2(3,3),B(NB,NB),A(NB,NB)
      M= N+1
      H= XL/N
      DO 1 I=1,M
      DO 1 J=1,M
    1 B(I,J)= 0.
      F2= H*H/24.
      DO 2 I=2,3
      DO 2 J=1,3
    2 B(I,J)= F2*B2(I,J)
      IF(N.EQ.2) GO TO 10
      N2= N/2
      DO 3 I=2,N2
      I1= 2*I
      I2= I1-1
      I4= I1+1
      DO 4 J=1,I2
      B(I1,J)= B(I2,J)+H*A(I2,J)
    4 B(I4,J)= B(I2,J)+2.*H*A(I2,J)
      B(I1,I2)= B(I1,I2)+B(2,1)
      B(I4,I2)= B(I4,I2)+B(3,1)
      B(I1,I1)= B(2,2)
      B(I1,I4)= B(2,3)
      B(I4,I1)= B(3,2)
    3 B(I4,I4)= B(3,3)
   10 RETURN
      END
```

```
      SUBROUTINE TRDGAL(N,XL,C2,C,NC,A,B)
      DIMENSION C2(3,3),C(NC,NC),A(NC,NC),B(NC,NC)
      M= N+1
      H= XL/N
      DO 1 I=1,M
      DO 1 J=1,M
    1 C(I,J)= 0.
      F3= H*H*H/240.
      H2= H*H/2.
      DO 2 I=2,3
      DO 2 J=1,3
    2 C(I,J)= F3*C2(I,J)
      IF(N.EQ.2) GO TO 10
      N2= N/2
      DO 3 I=2,N2
      I1= 2*I
      I2= I1-1
      I4= I1+1
      DO 4 J=1,I2
      C(I1,J)= C(I2,J)+H*B(I2,J)+H2*A(I2,J)
    4 C(I4,J)= C(I2,J)+2.*H*B(I2,J)+4.*H2*A(I2,J)
      C(I1 I2)= C(I1,I2)+C(2,1)
      C(I4,I2)= C(I4,I2)+C(3,1)
      C(I1,I1)= C(2,2)
      C(I1,I4)= C(2,3)
      C(I4,I1)= C(3,2)
    3 C(I4,I4)= C(3,3)
   10 RETURN
      END
```

```
      SUBROUTINE FRTGAL(N,XL,D2,D,ND,A,B,C)
      DIMENSION D2(3,3),D(ND,ND),A(ND,ND),B(ND,ND),C(ND,ND)
      M= N+1
      H= XL/N
      DO 1 I=1,M
      DO 1 J=1,M
    1 D(I,J)= 0.
      F4= H*H*H*H/720.
      H2= H*H/2.
      H3= H2*H/3.
      DO 2 I=2,3
      DO 2 J=1,3
    2 D(I,J)= F4*D2(I,J)
      IF(N.EQ.2) GO TO 10
      N2= N/2
      DO 3 I=2,N2
      I1= 2*I
      I2= I1-1
      I4= I1+1
      DO 4 J=1,I2
      D(I1,J)= D(I2,J)+H*C(I2,J)+H2*B(I2,J)+H3*A(I2,J)
    4 D(I4,J)= D(I2,J)+2.*H*C(I2,J)+4.*H2*B(I2,J)+8.*H3*A(I2,J)
      D(I1,I2)= D(I1,I2)+D(2,1)
      D(I4,I2)= D(I4,I2)+D(3,1)
      D(I1,I1)= D(2,2)
      D(I1,I4)= D(2,3)
      D(I4,I1)= D(3,2)
    3 D(I4,I4)= D(3,3)
   10 RETURN
      END
```