# TRIANGULAR FINITE ELEMENTS FOR PLATE BENDING

**Yalla Mussa Abushawash and Said Eshtew**

Mechanical Engineering Dept.
Al-Fateh University / Tripoli, Libya
yalla_zuara@hotmail.com
seshtewi@hotmail.com

**الملخص**

تم استخدام طريقة رباعية الشجرة (Quad-tree Method) في تقسيم متعدد الإضلاع إلي عناصر ثلاثية الأبعاد. تعتمد هذه الطريقة على التشكيل والتعديل الموضعي للشبكة التي إنشائها مسبقا. تبدأ العملية ببناء هيكل متزن لرباعية الشجرة بحيث لا يتجاوز عدد الرباعيات المجاورة اثنان. تتبع هذه المرحلة إنشاء العناصر المثلثية استنادا إلي النماذج المعرفة مسبقا. للحصول على النتائج المرجوة لعمليتي اتزان الهيكل وإنشاء العناصر المثلثية من المهم جدا تكوين نظام بحث فعال.

تم كتابة برنامج الحاسوب للعناصر المتناهية (Software of FE) والذي يحتوي على كل الخوارزميات اللازمة. النتائج المتحصل عليها تم دراستها وتحليلها للتأكد من مدى سلوك التقارب وأداء كل نوع من العناصر. العناصر المتناهية التي تم إنشائها أظهرت بيانات مقبولة وخاصة في العناصر المثلثية ذات الثمانية عشر درجة الحرية (DOF). أظهرت هذه العناصر اقتراب من الحل الأمثل مع زيادة العدد الكلي لدرجات الحرية.

**ABSTRACT**

Quad-tree Method is used to triangulate the polygonal two dimensional domains. The method is based on the deformation and local modification of an easily obtained grid which is started by building the balanced quad-tree construction that is bounded to be no leaf quad and can have more than two neighbours. Finally triangular elements were created using predefined templates. Efficient navigation system for the quad-tree construction is an important requirement to effectively perform the balancing condition and triangulation process.

The Finite element (FE) software developed contains all the required algorithms. Results were obtained for the convergence behaviour and the performance of each element. All elements show acceptable results especially for the Eighteen Degree of Freedom (DOF) triangular elements where the calculated maximum deflection converges to the exact solution as the total number of degrees of freedom increases.

**KEYWARDS**: Finite Element; Plate Bending; Triangular Finite Elements; Mesh Generation.

**INTRODUCTION**

In recent decades, the finite element method has become a mainstay for industrial engineering design and analysis. According to the applications (thermal, structural, mechanical, fluid, electromagnetic, etc. problems), numerical simulation by the finite element method requires the mesh data of the domain under consideration which can be reduced to an object or a set of objects. Whatever the case may be, the mesh must contain all useful information when considering the different steps in the numerical computation (geometry, definition of loads, computation of matrices, solution of systems, visualization of results, etc.) [1].

The derivation and development of plate bending elements has been a favourite topic of researchers and many different plate elements exist. Therefore a number of additional assumptions may be made for these theories. The reason that these elements receive so much attention is that there are different theories depending on the thickness of the plate [2].

The simulation of various physical phenomena (in chemistry, thermal analysis, electromagnetism, mechanics of solids, fluid mechanics, etc.) can be written in terms of partial differential equations which can be solved numerically using the finite element method [1]. The essence of this method consists of calculating approximate values of the solutions desired (temperatures, stresses, pressure, velocity, magnetic field, etc.). These values are computed at some points in the domain of interest, called the nodes. From this set of values, it is possible to derive the solution values at any position; this computational step is based on the use of chosen interpolation functions.

The numerical calculation requires, as a first step, the construction of a mesh of the domain where the problem is posed in order to define the nodes. This phase of pre-processing is very important. In the sense that the generation of a valid mesh in a domain with a complex geometry is not a trivial operation and can be very expensive in terms of the time required. On the other hand, it is crucial to create a mesh which is well adapted to the physical properties of the problem under consideration, as the quality of the computed solution is strongly related to the quality of the mesh [1].

It could be argued that plate type structures, being three-dimensional could be analyzed using three dimensional brick elements. However, because the thickness of the plate is much smaller than the other dimensions, aspect ratio error would be induced in the analysis resulting in poor accuracy of results. On the other hand, if the three-dimensional element were reduced in size so that the other dimensions of the element were comparable with the thickness, the error due to the aspect ratio would be eliminated. However, the number of elements required to achieve this, would result in a very large model which would require a massive amount of computer resources to run [2].

Meshing can be defined as the process of breaking up a physical domain into smaller sub-domains (elements) in order to facilitate the numerical solution of a partial differential equation. While meshing can be used for a wide variety of applications, the principal application of interest is the finite element method. Surface domains may be subdivided into triangle or quadrilateral shapes, while volumes may be subdivided primarily into tetrahedral or hexahedra shapes. The shape and distribution of the elements is ideally defined by automatic meshing algorithms.

At the inception of the finite element method, most users were satisfied to simulate vastly simplified forms of their final design utilizing only tens or hundreds of elements. Qualified preprocessing was required to subdivide domains into usable elements. Market forces have now pushed meshing technology to a point where users now expect to mesh complex domains with thousands or millions of elements with no more interactions than the push of a button.

The automatic mesh generation problem is that of attempting to define a set of nodes and elements in order to best describe a geometric domain, subject to various element size and shape criteria. The algorithm started by building the construction of the Quad-tree (the selected method for the base grid), then applying the balancing and warping conditions. The final mesh then obtained using some pre-defined triangular templates.

## TRIANGLE ELEMENT

### Shape Function

The three nodes triangle element used and the parent element can be represented by real element after transformation as shown in Figure (1). The shape function for this element can be expressed in terms of interpolation functions and its nodal coordinates,

$$x = \sum_{i=1}^{n} \Phi_i(\xi,\eta)x_i \tag{1}$$

And

$$y = \sum_{i=1}^{n} \Phi_i(\xi,\eta)y_i \tag{2}$$

As seen before, $\Phi_i$ is the interpolation function for the node *i* and n is the total number of nodes. For linear element, the nodes of the triangle element that should be included are only the three corner nodes, thus' the interpolation functions can be assumed as,

$$\phi_i = a_i + b_i\xi + c_i\eta \tag{3}$$

For (*i=1*) the interpolation function $\Phi_1$ will be unity at node *1* and zeros at nodes *2* and 3 thus,

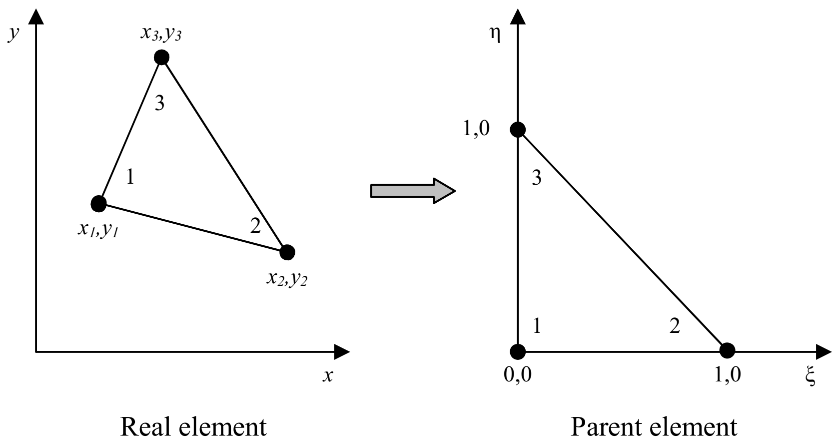


Figure 1: Parent and real triangle elements

$$\phi_1 = 1 - \xi - \eta \tag{4}$$

Similarly for (*i=2, 3*) the interpolations will be,

$$\phi_2 = \xi \tag{5}$$

$$\phi_3 = \eta \tag{6}$$

It's clear here that this geometric transformation functions are linear since we already had a linear transformation for the real coordinates, so its nature that could be only the first derivatives

$$\frac{\partial\Phi_1}{\partial\xi} = -1 \qquad \frac{\partial\Phi_2}{\partial\xi} = 1 \; \frac{\partial\Phi_3}{\partial\xi} = 0 \tag{7}$$

$$\frac{\partial \Phi_1}{\partial \eta} = -1 \qquad \frac{\partial \Phi_2}{\partial \eta} = 0 \qquad \frac{\partial \Phi_3}{\partial \eta} = 1 \tag{8}$$

A particular type of element is defined by a number of conditions that can be stated [4]:

- The element shape witch can be triangle, rectangle, etc…
- The coordinates of the interpolation nodes.
- The number of degrees of freedom.
- The definition of nodal variables.
- The polynomial basis of the approximation.
- The degrees of inter-element continuity that must be satisfied the number of displacement variables defined at each node will define the number of degrees of freedom per element that presents the order of polynomial that can be used to model the displacement variables within the element.

It's important here to notice that the criteria for convergence depend on the following element conditions. First the requirement of completeness which means that the displacement functions of the element must be able to represent the rigid body displacements and the constant strain state. On other hand the completeness means that no approximate lower-order terms of series should be omitted whilst higher-order terms are included. This is based on the fact that the trial functions must be capable of presenting both a constant value of the field variable and a constant first partial derivative.

Second the requirement of compatibility which means that the displacement within the elements and across the element boundaries must be continuous. Physically, compatibility ensures that no gaps occur between the elements when the assemblage is loaded.

Figure (2) shows the polynomial terms. These terms should be included to have complete polynomials in $x$ and $y$ for two dimensional analysis. It is seen that all possible terms of the form $x^a\ y^b$ are present, where $a+b=k$ and $k$ is the degree to which the polynomial is complete.

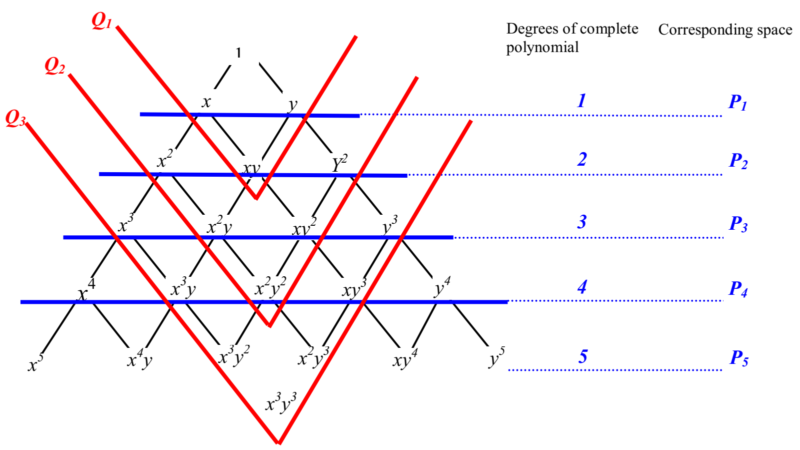


**Figure 2: Polynomial terms in two-dimensional analysis, Pascal triangle**

The above figure shows important notation for polynomial spaces. The spaces $P_k$ correspond to the complete polynomials up to degree $k$. In addition the figure shows also the polynomial spaces $Q_k$ which correspond to the 4-node, 9-node, and 16-node elements, referred to as Lagrangian functions. In practice, a frequently make observation that a satisfactory finite element analysis results have been obtained using incompatible (non-conforming) elements [4].

**Elements for $C^1$ Problems**

Constructing two-dimensional elements that can be used for problems requiring continuity of the field variable $w$ as will as its normal derivative $\frac{\partial w}{\partial n}$ along element boundaries, is far more difficult than constructing elements for $C^0$ continuity alone. To preserve $C^1$ continuity we must be sure that $w$ and $\frac{\partial w}{\partial n}$ are uniquely specified along the element boundaries by the degrees of freedom assigned to the nodes along a particular boundary. As point out in many studies, the difficulties arise from the following principles [5]:

- The interpolation functions must contain at least some cubic terms, because the tree nodal values $w$, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ must be specified at each corner of the element.
- For the nonrectangular elements $C^1$ continuity requires the specification of at least the six nodal values $w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$ and $\frac{\partial^2 w}{\partial x \partial y}$ at the corner nodes. For rectangular element with sides parallel to the global axes we need to specify at the corners nodes only $w$, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial^2 w}{\partial x \partial y}$.

It's sometimes very convenient to specify only $w$, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ at corners, but when this is done, it is impossible to have the second derivative continuity at the corner nodes.

Analysts first began to encounter difficulties in formulating elements for $C^1$ problems when they attempted to apply finite element techniques to plate bending problems. For such problems the displacement of the mid plane of the plate for Kirchhoff plate-bending theory is the field variable in each element and interelement continuity of the displacement and its slope is a desirable physical requirement. Also, since the functional for plate bending involves second-order derivatives, continuity of slope at element interfaces is a mathematical requirement because it ensures convergence as element size is reduced [5].

**Three Nodes, Nine DOF Triangular Element**

As we notice the plate bending element still capable to represent the element field variables with satisfactory amount of accuracy even for the elements of non-conforming type. This type of element shown in Figure (3) is one of the first elements used for plate bending problems and it shows a good convergence results. Nine degrees of freedom, three degrees of freedom per node $w$, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$.

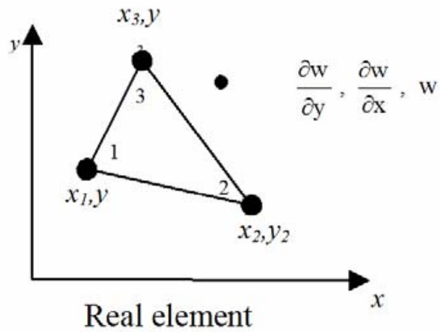


**Figure 3: three nodes nine degrees of freedom triangular element**

Here we meet the first difficulties, that there are no complete polynomials available to represent nine degrees of freedom (see Figure (3)); the complete cubic polynomial will have ten unknowns ($P_3$). One of the choices we have is that one of the terms $\xi^2\eta$ or $\xi\eta^2$ is omitted. In this case we use the following polynomial for our displacement interpolation weighing functions.

$$N=a_1+a_2\xi+a_3\eta+a_4\xi^2+a_5\xi\eta+a_6\eta^2+a_7\xi^3+a_8\xi^2\eta+a_9\eta^3 \tag{9}$$

To obtain the nine unknowns ($a_i$, $i=1, 2\ldots 9$) we need to define the condition of the nodal weighing values ($N_i$).

Finally the following expression could be obtained

$$Ga = f \tag{10}$$

In which $G$ is a $9x9$ matrix.

Indeed it's not easy to determine an explicit inverse of G and the stiffness expression, especially when the vector f contains some non-numerical terms or for the elements of higher order. This was evaluated in this study by using MatLab software. Notice that the matrix G will not change for the rest of displacement weighing functions, while the only change appear on the vector f. Substituting back into equation (10) and using Mat Lab

**Three Nodes Eighteen DOF Triangular Element**

This type of element shown in figure(3.4) is one of the conformal plate bending elements, the element is eighteen degrees of freedom, six degrees of freedom per node $w$, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$ and $\frac{\partial^2 w}{\partial x \partial y}$.

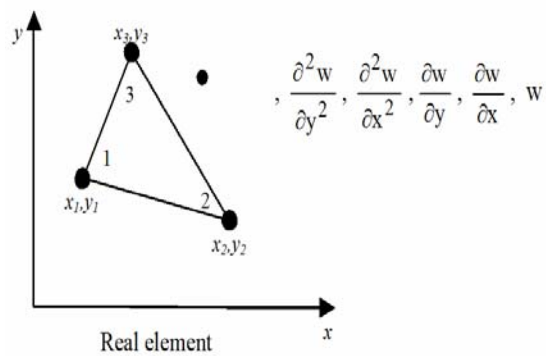


**Figure 4: three nodes eighteen degrees of freedom triangular element**

As we mentioned before the compatibility requirement for $C^1$ problems require the above six field variables to be continuous at the corner nodes. Here we meet the same complexity that we have seen in the nine DOF triangle elements, where also there are no complete polynomials available to represent eighteen DOF (see Figure (4)). The complete quartic polynomial ($P_4$) has only fifteen terms.

The following suggested polynomial is complete up to terms of fourth order and contains three terms of fifth order. The last three terms are chosen to force the normal derivative on each side to be cubic in ξ and η, on other hand the parabolic variation of the normal slope is not uniquely defined by the two end nodal values and hence resulted in the non-conformity [3].

$$\begin{aligned} N = {} & a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3 \\ & + a_{11}\xi^4 + a_{12}\xi^3\eta + a_{13}\xi^2\eta^2 + a_{14}\xi\eta^3 + a_{15}\eta^4 + a_{16}(\xi^5 - 5\xi^3\eta^2) \\ & + a_{17}(\xi^2\eta^3 - \xi^3\eta^2) + a_{18}(\eta^5 - 5\xi^2\eta^3) \end{aligned} \tag{11}$$

To see how the normal derivative is a cubic polynomial on each side of the real element, let us assume that the parent element shown in figure (3.5) to be the real element.

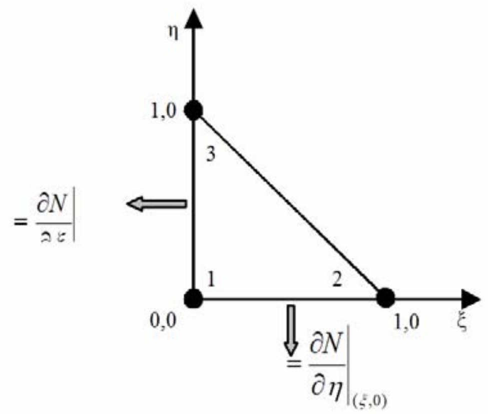


**Figure 5: Parent elements normal derivatives**

Similarly as we made previously to obtain the eighteen unknowns ($a_i$, $i=1,2,\ldots,18$) we need to define the condition of the nodal weighing values($N_i$)

The second derivatives $\frac{\partial^2 N_i}{\partial \xi^2}$, $\frac{\partial^2 N_i}{\partial \eta^2}$ and $\frac{\partial^2 N_i}{\partial \xi \partial \eta}$ will be evaluated.

In which the left hand matrix in equation (10) is the G(18x18) matrix was obtained.
Similarly, the rest vectors are expressed in the same manner by shifting the non-zero values six spaces and replaced by zeros for every six intervals. Substituting back into equation (10) and using MatLab.

**Six Nodes Eighteen DOF Triangular Element**

This type of element shown in Figure (6) it consists of six nodes, Three corner nodes and three mid side nodes, each node has three degrees of freedom $w$, $\frac{\partial w}{\partial x}$ and $\frac{\partial \mathrm{w}}{\partial \mathrm{y}}$.

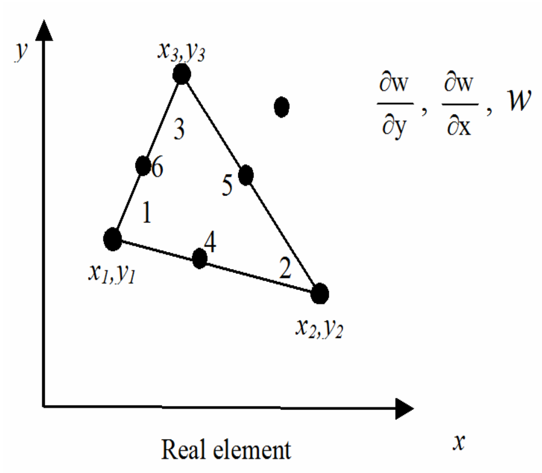


**Figure 6: six nodes eighteen degrees of freedom triangular element**

The suggested polynomial is the same polynomial used in the above element (equation (11)) which is complete up to terms of fourth order and contains three terms of fifth order. Similarly as we made previously to obtain the eighteen unknowns ($a_i$, $i=1,2,\ldots,18$) we need to define the condition of the nodal weighing values($N_i$). In a matrix form of equation (3.10) in which the left hand side is the G (18x18*)* matrix.

Similarly, the rest vectors are expressed in the same manner by shifting the non-zero values three spaces and replaced by zeros for every three intervals. Substituting back into equation (10) and using MatLab.

## RESULTS ANALYSIS AND DISCUSSION

The complete finite element package programme has been done before [6]. All algorithms were written in *Visual Basic*. The above meshing method and other pre-processing tools plus the required analysis functions are all made in one project. The programming part contains also some tools for the post-processing stage providing some features for the analyst to view the tabular and graphical results. The results discussion made for nine different cases. Those cases are modelled and analyzed using the three available elements. Results accuracy and convergence discussion was made for each of them.

Triangular elements are more versatile than rectangular elements because they can be used for the analysis of plates having various boundary shapes, such as skew or curved bridge decks [7]. At this point the use of triangular elements for the solution of plate bending problems is considered.

All elements show an acceptable results specially for the eighteen DOF triangular elements where the calculated maximum deflection converges to the exact solution as the total number of degrees of freedom increases. The Nine DOF element also presents results close to the exact solution but still incapable to insure the convergence criteria as we notice in the fixed edges square and circular plates.

In general the tabular outputs presents a reasonable cost of increasing the element degrees of freedom to eighteen, this can be even more clarified in the complicated curvature cases. We should notice here that the presented results could be more accurate with no more cost of the number of nodes and elements by involving the symmetric advantages witch is available in all the above cases.

**Table 1: Circular fixed edges plate $w_{max}$ error% for all elements**

| No. | Case | NN | NE | TDOF | Cal. $W_{max}$ | Ref. $W_{max}$ | [Error%] |
|---|---|---|---|---|---|---|---|
| | 3N 9DOF Tri Element | | | | | | |
| 1 | Circular fixed plate | 45 | 68 | 135 | -0.68524 | -0.66650 | 2.8113% |
| 2 | | 157 | 276 | 471 | -0.70582 | | 5.8982% |
| 3 | | 281 | 484 | 843 | -0.68280 | | 2.4450% |
| 4 | | 549 | 940 | 1647 | -0.75699 | | 13.5755% |
| 5 | | 601 | 1124 | 1803 | -0.69716 | | 4.5989% |
| 6 | | 821 | 1484 | 2463 | -0.67211 | | 0.8404% |
| | 3N 18DOF Tri Element | | | | | | |
| 1 | Circular fixed plate | 21 | 24 | 126 | -0.81328 | -0.66650 | 22.0215% |
| 2 | | 45 | 68 | 270 | -0.71016 | | 6.5506% |
| 3 | | 97 | 156 | 582 | -0.68905 | | 3.3833% |
| 4 | | 157 | 276 | 942 | -0.70612 | | 5.9443% |
| 5 | | 253 | 428 | 1518 | -0.68014 | | 2.0462% |
| 6 | | 281 | 484 | 1686 | -0.65775 | | 1.3131% |
| | 6N 18DOF Tri Element | | | | | | |
| 1 | Circular fixed plate | 65 | 24 | 195 | -0.62478 | -0.66650 | 6.2601% |
| 2 | | 125 | 52 | 375 | -0.69164 | | 3.7711% |
| 3 | | 157 | 68 | 471 | -0.64612 | | 3.0578% |
| 4 | | 349 | 156 | 1047 | -0.65065 | | 2.3780% |
| 5 | | 589 | 276 | 1767 | -0.65918 | | 1.0994% |
| 6 | | 933 | 428 | 2799 | -0.65239 | | 2.1172% |

## CONCLUSIONS

Triangle element for three nodes and Nine DOF is one of the non-conformal element type which still able to simulate the plate bending behaviour with an acceptable range of error. This simple element has also the low cost advantage since its only Nine DOF element. For more complicated elements we derive three nodes Eighteen DOF triangular elements, this element insures the Compatibility and Completeness conditions which classifies the element as one of the conformal type elements.

Numerical results for the maximum deflection in a number of cases made in this study. By using different number of Total Degrees Of Freedom (TDOF) results of deflection were all converge to the theoretical exact solution. The last element derived was a higher order triangle element of six nodes and Eighteen DOF, the element has the same interpolation function with the same nodal DOF used in the first element. However, the cost of increasing the element Degrees Of Freedom and their trial functions order was reasonable.

## EFERENCES

[1] P.L. George (1991), "Automatic Mesh Generation, Application to Finite Element Method", John Wiley and Sons.

[2] Rae S Gordon (1993), "ITTI Finite Element Training Project Static I: Plate & Shell Analysis", MCC UMIST Support Unit
[3] Dhatt, G. and Touzot, G. (1984), "The Finite Element Method Displayed", John Wiley and Sons.
[4] Klaus-Jurgen Bathe (1996), "Finite Element Procedures", Prentice Hall, New Jersey.
[5] Kenneth H. Huebner, Earl A. Thornton (1982), "The Finite Element Method for Engineers", Second Edition, John Wiley and Sons.
[6] Said Eshtew, Yalla Mussa Abushawash (2004), "Discretization and Analysis in Finite Element for Plate Bending", MSc Thesis Mechanical Engineering and Industrial Faculty of Engineering Al-Fath University Tripoli Libya
[7] K. C. Rocky, H. R. Evans, D. W. Griffiths, D. A. Nethercot (1990), "The Finite Element Method, A Basic Introduction", Crosby Lockwood Staples, London.