AN EFFICIENT FINITE ELEMENT OF TORSIONAL DYNAMIC ANALYSIS FOR OPEN THIN-WALLED BEAMS UNDER TORSIONAL EXCITATIONS

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الملخص

تم تطوير عنصر عارضة محدود فائق التقارب (super-convergent finite beam element) باستخدام نظرية العناصر المتناهية من أجل التحليل الديناميكي المقترن للالتواء والفتل للعارضات رقيقة الجدار ذات مقاطع عرضية مفتوحة متماثلة والمعرضة لعزوم الالتواء والفتل التوافقية المختلفة .تم اشتقاق المعادلات الديناميكية للحركة والشروط الحدودية ذات الصلة للاستجابة المزدوجة الالتوائية والفتل في دراسة سابقة. تعتمد صيغة العناصر المحدودة على نظرية العارضة العامة لفلاسوف - تيموشينكو (Vlasov-Timoshenko beam theory) والتي تأخذ في الحسبان تأثيرات تشوه القص (shear deformation) بسبب الالتواء الغير منتظم، كما أنها تلتقط تأثيرات القوى الاستاتيكية المحورية الثابتة على الترددات الالتوائية الطبيعية والاستجابات الاستاتيكية والديناميكية. تم اشتقاق مجموعة دوال الشكل الدقيقة (exact shape functions) بناءً على الحل الدقيق (exact solution) للمعادلات الديناميكية للالتواء والفتل بحيث تُستخدم لصياغة عنصر العارضة المحدود الحالي والذي يتمتع بعقدتين وأربع درجات من الحرية .تم بنجاح استخدام عنصر العارضة المحدود المطور بهذه الدراسة للحصول على الاستجابات الديناميكية المزدوجة للالتواء والفتل للعارضات رقيقة الجدار ذات مقاطع عرضية مفتوحة متماثلة والمعرضة لعزوم الالتواء والفتل التوافقية. كذلك استخدم العنصر المحدود لاستخراج الترددات الطبيعية الملتوية وأشكال النسق (torsional natural frequencies and mode shapes) للنظام من التحليل الديناميكي. تم إثبات خلو عنصر العارضة المحدود الحالي من الأخطاء الناتجة عن التجزئة التي تحدث لحلول العناصر المحدودة التقليدية. تم التحقق من قابلية تطبيق عنصر العارضة المحدود من خلال عدد من أمثلة عددية. تبين النتائج العددية التي تم الحصول عليها من الحل الحالي أنه يوجد اتفاق ممتاز(excellent agreement) مع حلول العناصر المحدودة الأخرى بفارق صغير في التكلفة الحسابية والنمذجة.

ABSTRACT

A super-convergent finite beam element formulation is developed for the torsional-warping dynamic coupled analysis of thin-walled open doubly symmetric beams under various harmonic torsional and warping moments. The dynamic equations of motion and related boundary conditions for torsional warping coupled response were derived in previous study. The finite element formulation is based on a generalized Vlasov-Timoshenko beam theory, and accounts for shear deformation effects due to non-uniform warping. It is also capturing the effects of axial constant static forces on the natural torsional frequencies, quasi-static and steady state dynamic responses. A family of shape functions is developed based on the exact solution of the coupled equations and are then used to formulate a beam finite element. The new two-nodded beam element with four degrees of freedom per element successfully captured the coupled torsional-warping quasi-static and steady state dynamic responses of open thin-walled beams under various harmonic torsional and warping moments. It is also used to extract the coupled torsional-warping natural frequencies and mode shapes from the dynamic analysis of the structural member. The present beam element is demonstrated to be free from discretization errors occurring in conventional finite element solutions. The applicability of the finite beam element is verified through several numerical examples. The numerical results based on the present finite element solution are found to be in excellent agreement with those based on exact and Abaqus finite element solutions available in the literature at a small fraction of the computational and modelling cost involved.

KEYWORDS: Exact shape functions; Torsional-Warping Coupled Response; Super-Convergent Finite Element.

INTRODUCTION AND OBJECTIVE

Thin-walled members are commonly used in the design of many structural components in aerospace structures, steel building construction, steel bridges, ship and marine structural frames, truck frames, and so forth. In such applications, thin-walled beams subjected to cyclic harmonic torsional excitations are prone to fatigue failures. Under these harmonic torsional loads, the total response of a thin-walled beam is a combination of two components; (a) a transient torsional response which is initiated at the beginning of the excitation, and (b) a steady state torsional response which is sustained for a longer time. The transient torsional response attenuates quickly due to damping and is thus of no importance for fatigue design. In contrast, the sustained steady state component of the torsional response is of major importance for fatigue design and is the subject of the present study. Within this context, the present paper aims at developing an efficient finite element solution which captures and isolates the steady state torsional-warping coupled dynamic response of open thin-walled doubly symmetric beams. The present finite beam element solution is also able to capture the effect of axial constant tensile and compressive forces on the quasi-static, steady state torsional dynamic responses and torsional eigen-frequencies and eigen-modes of the system.

LITERATURE REVIEW ON ANALYTICAL SOLUTION

Thin-walled beam theories which capture warping effects include the works of [1], and [2]. Reference [1] developed a general theory for isotropic thin-walled beams with open and closed cross-sections which captures the warping effects. Compared to the typical Saint Venant torsion theory, the Vlasov theory introduced the rate of change of the torsional rotation angle as a measure of warping deformation, which leads to an additional straining action, the bimoment. The Vlasov torsion formulation is based on two fundamental kinematic assumptions: (i) the cross section of a member remains undeformed (or rigid) after deformation, and (ii) the shear strain in the middle surface is neglected. In other words, Vlasov torsion theory for thin-walled beams considers the warping stiffness of the beam cross section but neglects the shear deformation effects at the middle surface. Reference [2] extended the theory of Vlasov to account for the additional through-thickness secondary warping for beams with open and closed cross-sections. In a similar theory, [3] independently developed a theory for isotropic beams with open cross-sections in which the shear deformation effects are included.

Several publications based on the analytical solutions of the static analysis and free torsional vibration of open thin-walled beams with doubly symmetric cross-section, considering the warping deformation of the cross-section and by including/excluding the axial static effects are investigated by some publications. Among them, [4] investigated the free torsional vibration of doubly symmetric long thin-walled beams of open section. In his formulation, the warping effect of the cross-section on the natural frequencies and normal mode shapes are determined for thin-walled bars with various end conditions. Based on dynamic stiffness matrix approach, [5] investigated the free torsional vibration and buckling of doubly symmetric open thin-walled beams subjected to an axial static compressive load and resting on continuous elastic foundation. Reference [6] derived the closed-form solutions for the torsional analysis of thin-walled beams under various twisting moments and boundary conditions. Reference [7] developed a boundary element solution for the general linear elastic non-uniform torsion problem of homogeneous and composite prismatic bars of arbitrary cross section subjected to various twisting moments. Reference [8] developed the dynamic stiffness matrix formulation for computing the natural torsional frequencies of elastically restrained doubly symmetric thin-walled I-beams resting on Winkler-type continuous elastic foundation. In their formulation, the analytical solution is developed by including the effects of warping deformation and excluding the longitudinal inertia and shear deformation effects. Reference [9] presented an improved thin-walled beam theory considering the transverse shear deformation due to the shearing force and restrained warping and the coupled effect between these two shear deformations by introducing Vlasov's assumption and applying Hellinger- Reissner principle. Reference [10] developed an analytical method for the torsion of open thin-walled beams with effect of shear deformation by assuming that the shear stress was constant along the beam length. Based on postulated stress field, [11] developed a theory for the torsional static analysis of open steel thin-walled beams of general cross sections which accounted for shear deformation effects. References [12, 13] presented a beam theory with a non-uniform warping including the effects of torsion and shearing forces. Based on Vlasov's and Benscoter's theories, [14] presented an exact solution of non-uniform torsion for thin-walled elastic beams with asymmetric cross-section. Based on the boundary element method, [15] developed a non-uniform torsion theory of doubly symmetrical arbitrary cross-section including secondary torsional moment deformation effect. Reference [16] developed an exact closed form solution for the steady state torsional dynamic response of open thin-walled beams of doubly symmetric cross-sections subjected to various harmonic torsional moments. Their formulation was based on generalized Timoshenko-Vlasov beam theory in which the transverse shear deformation induced by non-uniform warping is incorporated. Reference [17] developed a first-order torsion theory based on Vlasov theory for restrained torsion of open thin-walled beams. The theory captured the warping deformation and restrained shear deformation of the cross-section. Reference [18] presented the static and dynamic analyses of the geometrically linear or nonlinear, elastic or elastic-plastic non-uniform torsion problems of bars of constant or variable arbitrary cross section subjected to arbitrarily distributed or concentrated twisting and warping moments along the bar axis. Based on the classical Vlasov's theory, [19] developed a theory for torsion of thin-walled beams with influence of shear deformation for open cross-sections with single and double axes of symmetry and under various torsional loads. Based on Vlasov beam theory, [20] formulated an analytical solution for the dynamic response analysis of doubly symmetric thin-walled I-beams under harmonic flexural and torsional loadings. Their solution considers the effect of warping deformation of the cross-section. From Saint-Venant and non-uniform torsional deformations, [21] investigated the effect of constant thermal gradient on the torsional natural frequencies of open thin-walled pre-stressed beams. According to Vlasov beam theory, [22] derived the closed form solutions for the coupled flexural-torsional dynamic response of thin-walled beams with mono-symmetric cross-sections under harmonic excitations. Their formulation takes into consideration the effects of translational and rotary inertia, warping deformation and flexural-torsional coupling due to cross section mono-symmetry. Reference [23] derived an analytical solution of torsional vibrations of prismatic thin-walled beams for different boundary conditions and various external excitation of torsional moment. Their solution is based on the Vlasov beam theory where the warping deformation of the cross section is included. Recently, [24] extended the work of [17] to formulate the exact closed-form solution by investigating the effect of axial static tensile and compressive forces on the coupled torsional-warping static and dynamic responses of open doubly symmetric thin-walled beams subjected to various dynamic torsional excitations. More recently, [25] developed an exact closed-form solution for the torsional static analysis of open thin-walled doubly symmetric beams under various torsional and warping moments. Their formulation based on generalized Vlasov-Timoshenko beam theory which considers the effect of warping deformation of the cross-section due to shear deformation.

LITERATURE REVIEW ON FINITE ELEMENT FORMULATION

In general, finite element formulations are based on three categories of shape functions: (1) approximate polynomial interpolation functions, (2) shape functions based on the exact solution of the static equilibrium equations, and (3) shape functions based on the exact solution of the dynamic equations of motion. Formulations based on the approximate shape functions are most common and are included in the work of [26-33], and recently [34]. Using the approximate interpolation functions, reference [26] used the finite element method to study the torsional vibration of long thin-walled beams of open section resting on the elastic foundation. By utilizing Galerkin-based finite element method, [27] studied the free torsional vibration of linearly tapered cantilever I-beams. Reference [28] developed a finite element for the analysis of thin-walled open members under constant transverse loads. Their formulation was based on assumed linear and cubic displacement shape functions, in conjunction with an implicit self-starting unconditionally stable integration scheme. Reference [29] developed a finite element for the analysis of thin-walled beams with arbitrary open cross-sections. Finite element formulations including shear deformation effects include the work of [30] who formulated an isoparametric element to capture the coupled flexural-torsional free vibration of asymmetric thin-walled shear deformable beams. References [31,32] study the coupled flexural-torsional composite members to incorporate the shear deformation effects in a finite element formulation based on one-dimensional shear-deformable finite beam element using linear and cubic Hermite shape functions. Reference [33] formulated the governing differential equation for non-uniform torsion of thin-walled beams with open/closed cross-sections according to the theory of second-order torsional warping. Their formulation captured the effect of variable axial force and secondary torsion-moment deformation effect on the beam deformations due to torsional warping. In addition, the transfer matrix method is derived to develop a finite beam element with two nodes for static and dynamic analyses of beams. Recently, based on Saint-Venant and non-uniform torsional deformations, [34] developed a finite element method based on Vlasov theory to analyze the stress state induced due to bimoments of open thin-walled bars.

Finite element solutions based on the exact solution for the static equilibrium equations such as the work of [35-38], and recently [39]. Their formulations have the advantage of avoiding locking problems, which could arise in some of the solutions based on polynomial interpolation functions. In [35], a finite element is developed for the coupled free vibrations analysis of thin-walled beams. The formulation incorporated warping effects and was based on shape functions derived based on the solution of static equilibrium equations. Reference [36] formulated a finite element formulation for the coupled bending-torsional dynamic behavior of thin-walled beams of asymmetric cross-sections.

The interpolation functions adopted were based on the homogenous solutions of static differential equations of equilibrium and were used to derive the stiffness and mass matrices of the beam element in the finite element formulation. Reference [37] developed a finite beam element formulation used the exact static solution of torsional analysis of thin-walled beams with open cross-sections based on St. Venant and Vlasov theories. Based on a generalized Timoshenko-Vlasov thin-walled beam theory, [38] developed a super-convergent finite beam element solution for the coupled flexural-torsional analysis of monosymmetric thin-walled open members under general static forces. The two-noded finite element with four degrees of freedom per node based on shape functions which exactly satisfy the homogeneous form of the equilibrium static coupled equations is developed to fully capture the effects of warping stiffness, shear deformation, and established the torsional-flexural coupling. Lately, [39] used the exact homogeneous solutions for torsional rotation and warping deformation functions to formulate an exact finite beam element solution of torsional-warping coupled static response of open thin-walled doubly symmetric beams.

Finite-element solutions based on the exact solution of the dynamic equations of motion include the work of [40- 42]. Based on Vlasov beam theory, [40] formulated a super-convergent two-noded finite beam element solution for the dynamic response analysis of doubly symmetric thin-walled I-beams under harmonic flexural and torsional loadings. The formulation considers the effect of warping deformation of the cross-section. In their finite element formulations, a family of exact shape functions for torsional rotation and warping deformation were developed based on the exact homogeneous solutions of the governing torsional equations. In another publication, [41] developed an exact finite element formulation for the coupled flexural-torsional dynamic response of open monosymmetric thin-walled beams. The beam element based on Vlasov beam theory assumptions captures the effects of Saint Venant and warping torsion translational and rotary inertia and the coupling between bending and torsion. Reference [42] formulated a super-convergent two-nodded finite beam element based on the exact shape functions which satisfy the exact homogeneous solution of the governing torsional equation to investigate the quasi-static and dynamic analyses for the torsional vibration of shafts subjected to various harmonic twisting moments.

The finite element formulations based on approximate shape functions involve spatial discretization errors, and thus require fine meshes to converge to the actual solution. In contrast, the finite element formulations based on exact solutions offer two advantages: (1) they eliminate discretization errors arising in conventional interpolation schemes and converge to the solution using a minimal number of degrees of freedom; and (2) they lead to elements that are free from shear locking. Within this context, the present paper aims to develop an efficient finite beam element solution for the torsional-warping coupled dynamic analysis of thin-walled beams with doubly symmetric open sections subjected to harmonic torsional and warping moments and axial static force. The formulation sought is based on exact shape functions which exactly satisfy the coupled torsional-warping field equations and captures shear deformation effects caused by warping. The present paper differs from [24] as the previous paper achieved the exact closed-form solution of the torsional-warping coupled response of open thin-walled beams subjected to torsional and warping harmonic moments, while the present paper is an extension of the previous paper and develops an efficient finite beam element solution that depends on the exact shape torsional and warping deformation functions which exactly satisfied the solution of the coupled field equations derived in [24].

MATHEMATICAL MODEL AND GOVERNING EQUATIONS

Consider a linearly elastic, homogeneous, isotropic open thin-walled beam subjected to distributed harmonic torsional and warping moments and axial static force, undergoing coupled torsional-warping linear vibrations. A generalized Timoshenko bending, and Vlasov torsion beam theories are used to derive the governing differential equations of motion and an efficient finite beam element solution based on the exact shape functions is developed. The thin-walled beam is referenced to a right-handed rectangular coordinates system (X, Y, Z), where the axis Z is the longitudinal axis of the beam, while X and Y are the principal axes of the cross-section passing through the section centroid C. Figure (1) shows the coordinate systems and geometry of the open thin-walled cross-section, where L is the length of the beam. The two governing differential coupled equations of the open thin walled doubly symmetric beam were derived in previous studies [24] are given as follows:

$$\rho A r_o^2 \ddot{\theta}_z(z,t) - (GJ + GD_{ww} - P_{zo} r_o^2)\theta_z''(z,t) - GD_{ww}\psi'(z,t) = m_z(z,t) \quad (1)$$

$$GD_{ww}\theta_z'(z,t) + \rho I_w \psi(\ddot{z},t) - EI_w\psi''(z,t) + ED_{ww}\psi(z,t) = -m_w(z,t) \quad (2)$$

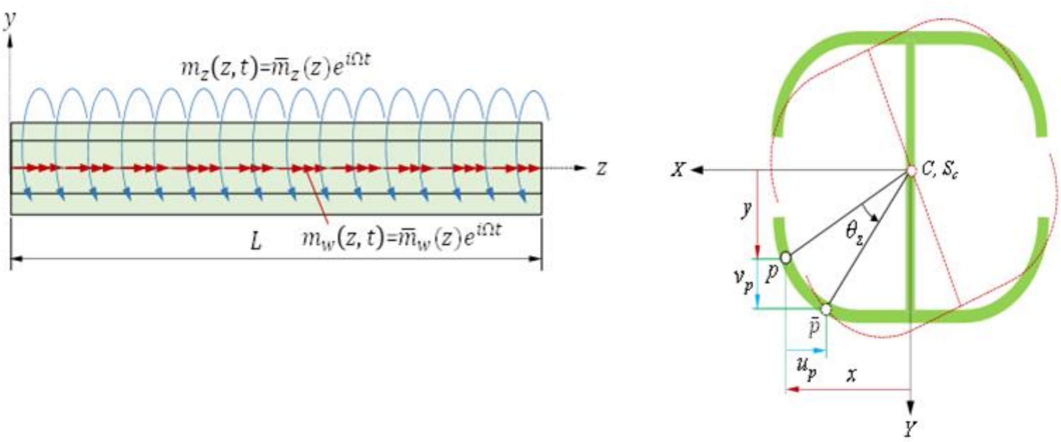


Figure 1: Open Thin walled doubly symmetric beam subjected to various dynamic torsional and warping moments

where $\theta_z(z,t)$ is the torsional rotation of the cross-section, $\psi(z,t)$ is a function which characterizes the magnitude of the warping deformation, $\omega(s)$ is the warping function of the open cross-section is defined by: $\omega(s) = \int_s \ h(s)ds$, in which $h(s)$ is the perpendicular distance from the shear center S_c to the tangent to the mid-surface at point $p(x,y)$, $r_o^2 = (I_{xx} + I_{yy})/A$ is the polar radius of gyration about the shear centre, ρ is the material density, E is the modulus of elasticity, G is the shear modulus, J is the St. Venant torsional constant, and A is the cross-sectional area, I_w is the warping constant, Ω is the circular exciting frequency of the applied torsional moments, where $A, I_{xx}, I_{yy}, I_w, D_{ww} = \int_A [1, y^2, x^2, \omega^2, \ h^2] \, dA$. All primes denote derivatives with respect to space coordinate z while dots denote the derivatives with respect to time. In equations (1,2), P_{zo} is the axial static force, $m_z(z,t)$ is the harmonic distributed torsional moment, $m_w(z,t)$ is the harmonic distributed warping moments (i.e., bimoments) applied along beam axis (Figure 1).

The above equations are applicable to thin-walled beams having doubly symmetric open cross-sections and are restricted to the torsional-warping coupled response of open section thin-walled beams. In this formulation, the shear deformation effects induced by warping (i.e., non-uniform torsion) at the middle surface of the cross-section are assumed non-zero and are characterized by a generalized displacement function multiplied by the sectorial coordinate (the reader is referred to the previous paper [24] for the basic assumptions of the formulation, description of the kinematics and the solution of the problem).

Expressions for Applied Moments and Functions

The open thin-walled beam is assumed to be subjected to the applied harmonic twisting and warping moments within the member:

$$m_z(z,t), m_w(z,t) = [\bar{m}_z(z), \bar{m}_w(z)]e^{i\Omega t} \tag{3}$$

Under the given harmonic torsional moments and in the absence of damping, the torsional rotation and warping deformation functions corresponding to the steady-state component of the dynamic response are assumed to take the form:

$$\theta_z(z,t), \psi(z,t) = [\bar{\Theta}_z(z), \bar{\Psi}(z)]e^{i\Omega t} \tag{4}$$

in which $i = \sqrt{-1}$ is the imaginary constant, $\bar{\Theta}_z(z)$ and $\bar{\Psi}(z)$ are the amplitude space functions for torsional rotation, and warping deformation, respectively. Because the present formulation is intended to capture only the steady-state dynamic response of the system, the torsional rotation and warping deformation functions postulated in equation (4) disregard the transient component of the dynamic response.

Solution of Torsional-Warping Coupled Equations

From the harmonic expressions in equations (3,4) and by substituting into equations (1,2), one obtains the coupled torsional-warping dynamic equations:

$$\begin{bmatrix} (P_{zo}r_o^2 - GJ - GD_{ww})\mathcal{D}^2 - \rho A r_o^2 \Omega^2 & -GD_{ww}\mathcal{D} \\ -GD_{ww}\mathcal{D} & \rho I_w \Omega^2 - GD_{ww} + EI_w\mathcal{D}^2 \end{bmatrix}_{2\times 2} \begin{Bmatrix} \bar{\Theta}(z) \\ \bar{\Psi}(z) \end{Bmatrix}_{2\times 1} = \begin{Bmatrix} \bar{m}_z(z) \\ \bar{m}_w(z) \end{Bmatrix}_{2\times 1} \tag{5}$$

in which $\mathcal{D}$ is the differential operator, i.e., $\mathcal{D} \equiv d/dz$and $\mathcal{D}^2 = d^2/dz^2$. The homogeneous solution of the coupled torsional-warping equations in (5) was obtained in previous study [24] as:

$$\{\Phi(z)\}_{2\times 1} = [\bar{G}]_{2\times 4}\,[E(z)]_{4\times 4}\{\bar{A}\}_{4\times 1} \tag{6}$$

in which,

$\langle \Phi(z) \rangle_{1\times 2} = \langle \bar{\Theta}_z(z) \quad \bar{\Psi}(z) \rangle_{1\times 2}$, $[E(z)]_{4\times 4} = Diag\,[e^{\beta_1 z} \quad e^{\beta_2 z} \quad e^{\beta_3 z} \quad e^{\beta_4 z}]_{4\times 4}$, the unknown integration vector is $\langle \bar{A} \rangle_{1\times 4} = \langle A_1 \quad A_2 \quad A_3 \quad A_4 \rangle_{1\times 4}$, and $[\bar{G}]_{2\times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{bmatrix}_{2\times 4}$, where $\mu_i = -\left(\frac{\rho A \Omega^2 r_o^2 + G(J + D_{ww})\beta_i^2}{GD_{ww}\beta_i}\right) = \left(\frac{GD_{ww}\beta_i}{EI_w\beta_i^2 + (\rho I_w \Omega^2 - GD_{ww})}\right)$.

It is noted that, all four roots ($m_i = \beta_i$ for $i = 1,2,3,4$) are distinct and are given by $\beta_{1,2} = m_{1,2} = \pm\sqrt{\propto + \lambda}$, and $\beta_{3,4} = m_{3,4} = \pm i\sqrt{\propto + \lambda}$, where

$$\alpha = \frac{[\![\rho I_w \Omega^2 [G(J + D_{ww}) - r_o^2(P_{zo} - EA)] - GD_{ww}(GJ - P_{zo}r_o^2)]\!]}{2EI_w[G(J + D_{ww}) - P_{zo}r_o^2]}$$

$$\lambda = \left[\left(\frac{\rho I_w \Omega^2 [G(J + D_{ww}) - r_o^2 (P_{zo} - EA)] - G D_{ww} (GJ - P_{zo} r_o^2)}{2 E I_w [G(J + D_{ww}) - P_{zo} r_o^2]} \right)^2 - \frac{\rho A \Omega^2 r_o^2}{E I_w} \left(\frac{\rho I_w \Omega^2 - G D_{ww}}{G(J + D_{ww}) - P_{zo} r_o^2} \right) \right]^{1/2}$$

Formulation of Exact Finite Element

The proposed finite beam element is developed for the coupled torsional-warping dynamic response of open thin-walled beams under various harmonic torsional and warping moments. The proposed two-nodded finite beam element having four degrees of freedom per element is developed (Figure 2). A set of exact shape functions that exactly satisfy the homogeneous solution of the coupled field equations in [24] is used to formulate the exact stiffness and mass matrices and load potential energy vector for the beam element.

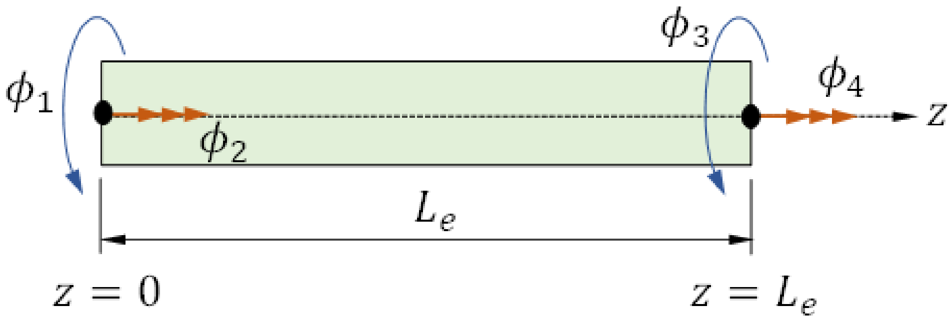


Figure 2: Two-nodded beam element for torsional-warping coupled response

Expressions of Exact Shape Functions

To relate the torsional rotation $\bar{\Theta}_z(z)$ and warping deformation $\bar{\Psi}(z)$ functions to the nodal torsional and warping deformation, the vector of integration constants $\{\bar{A}\}_{4\times1}$is expressed in terms of nodal torsional and warping displacements $\langle d_e \rangle_{1\times4} = \langle \phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4 \rangle_{1\times4}$by enforcing the conditions $\bar{\Theta}_z(0) = \phi_1$, $\bar{\Psi}(0) = \phi_2$, $\bar{\Theta}_z(L_e) = \phi_3$ and $\bar{\Psi}(L_e) = \phi_4$, where L_e is the beam element length, yielding:

$$\{d_e\}_{4\times1} = \begin{Bmatrix} \{\Phi(0)\}_{2\times1} \\ \{\Phi(L_e)\}_{2\times1} \end{Bmatrix}_{4\times1} = \begin{bmatrix} [\bar{G}]_{2\times4} \, [E(0)]_{4\times4} \\ [\bar{G}]_{2\times4} \, [E(L_e)]_{4\times4} \end{bmatrix}_{4\times4} \{\bar{A}\}_{4\times1} = [\, S \,]_{4\times4} \{\bar{A}\}_{4\times1} \tag{7}$$

From equation (7), by substituting into equation (6), one obtains:

$$\{\Phi(\mathrm{z})\}_{2\times1} = [\, E(z) \,]_{2\times4} \, [S]^{-1}_{4\times4} \{d_e\}_{4\times1} = [\, H(z) \,]_{2\times4} \, \{d_e\}_{4\times1} \tag{8}$$

in which $[\, H(z) \,]_{2\times4} = [\, H_{1,j}(z) \quad H_{2,j}(z) \,]_{2\times4} = [\, E(z) \,]_{2\times4} \, [S]^{-1}_{4\times4}$ is a matrix of eight shape functions for torsional rotation and warping deformation for steady state dynamic response. It is obvious that, equation (8) provided the exact shape functions that exactly satisfy the homogeneous solution of the torsional-warping steady state dynamic coupled equations are dependent on the beam length, exciting frequency, and cross-section properties.

Energy Expressions in Terms of Nodal Torsional Displacements

The variation of kinetic energy, strain energy and work done due to applied harmonic torsional and warping moments and axial static force are obtained in terms of nodal degrees of freedom by substituting equation (7) into equations (9-12) given in [24] as:

$$\delta T = -\langle \delta d_e \rangle_{1\times 4} \left(\Omega^2 \int_0^{L_e} [[H(z)]_{4\times 2}^T [Z_m]_{2\times 2} [H(z)]_{2\times 4}] dx \right) \{d_e\}_{4\times 1}\, e^{i\Omega t} \tag{9}$$

$$\begin{aligned} \delta U = \langle \delta d_e \rangle_{1\times 4} \Bigg(\int_0^{L_e} & [[H'(z)]_{4\times 2}^T [Z_k]_{2\times 2} [H'(z)]_{2\times 4} \\ & + [H_d(z)]_{4\times 2}^T [Z_d]_{2\times 2} [H_d(z)]_{2\times 4}] dz \Bigg) \{d_e\}_{4\times 1}\, e^{i\Omega t} \end{aligned} \tag{10}$$

$$\delta V_1 = -\langle \delta d_e \rangle_{1\times 4} \left(\int_0^{L_e} [H(z)]_{4\times 2}^T \{Q_F\}_{2\times 1} dz + [[H(z)]_{4\times 2}^T \{Q_m\}_{2\times 1}]_0^{L_e} \right) e^{i\Omega t} \tag{11}$$

$$\delta V_2 = \langle \delta d_e \rangle_{1\times 4} \left(\int_0^{L_e} [H_p(z)]_{4\times 2}^T [Z_p]_{2\times 2} [H_p(z)]_{2\times 4}\, dz \right) e^{i\Omega t} \tag{12}$$

where $[Z_m]_{2\times 2} = Diag[\rho A r_o^2 \quad \rho I_w]_{2\times 2}$, $[Z_k]_{2\times 2} = Diag[GJ \quad EI_w]_{2\times 2}$, $[Z_d]_{2\times 2} = \begin{bmatrix} GD_{ww} & GD_{ww} \\ GD_{ww} & GD_{ww} \end{bmatrix}_{2\times 2}$, $[Z_p]_{2\times 2} = Diag[P_{zo} r_o^2 \quad 0]_{2\times 2}$, $[H'(z)]_{4\times 2}^T = [H'_{1,j}(z) \quad H'_{2,j}(z)]_{2\times 4}^T$, $[H_d(z)]_{2\times 4} = [H'_{1,j}(z) \quad H_{2,j}(z)]_{2\times 4}$, $[H_p(z)]_{2\times 4}^T = [H'_{1,j}(z) \quad 0]_{2\times 4}^T$, $\langle Q_F \rangle_{1\times 2} = \langle \bar{m}_z(z) \quad \bar{m}_w(z) \rangle_{1\times 2}$, and $\langle Q_m \rangle_{1\times 2} = \langle [\bar{M}_z(z)]_0^{L_e} \quad [\bar{M}_w(z)]_0^{L_e} \rangle_{1\times 2}$.

in which $\bar{M}_z(z)$ and $\bar{M}_w(z)$ are the harmonic end twisting and warping moments applied at beam ends $(i.e., z = 0, L)$.

Matrix Formulation

The variational form of the Hamilton's principle is expressed as:

$$\int_{t_1}^{t_2} \delta T dt - \int_{t_1}^{t_2} (\delta U + \delta V) dt = 0 \tag{13}$$

From equations (9-12), by substituting into Hamilton's variational principle in equation (13), one obtains:

$$([K_e]_{4\times 4} - \Omega^2 [M_e]_{4\times 4}) \{d_e\}_{4\times 1} = \{F_e\}_{4\times 1} \tag{14}$$

in which, the stiffness matrix for beam element $[K_e]_{4\times 4}$ is given by:

$$[K_e]_{4\times 4} = \int_0^{L_e} [[H'(z)]_{4\times 2}^T [Z_k]_{2\times 2} [H'(z)]_{2\times 4} + [H_d(z)]_{4\times 2}^T [Z_d]_{2\times 2} [H_d(z)]_{2\times 4}] dz \tag{15}$$

The mass matrix for beam element $[M_e]_{4\times 4}$ is given by:

$$[M_e]_{4\times 4} = \int_0^{L_e} [H(z)]_{4\times 2}^T [Z_m]_{2\times 2} [H(z)]_{2\times 4}\, dz \tag{16}$$

The element load vector $\{F_e\}_{4\times 1}$ is given by:

$$\begin{aligned} \{F_e\}_{4\times 1} = \int_0^{L_e} & \left([H(z)]_{4\times 2}^T \{Q_F\}_{2\times 1} + [H_p(z)]_{4\times 2}^T [Z_p]_{2\times 2} [H_p(z)]_{2\times 4} \right) dz \\ & + [[H(z)]_{4\times 2}^T \{Q_m\}_{2\times 1}]_0^{L_e} \end{aligned} \tag{17}$$

NUMERICAL RESULTS AND DISCUSION

In this section, several examples for thin-walled open beams of doubly symmetric cross sections subjected to various harmonic torsional and warping moments and different boundary conditions are presented to demonstrate the validity, accuracy, and applicability of the present finite beam element formulation. While the above formulation provides the dynamic response under harmonic torsional loads, it can also (*i*) capture the quasi-static response under harmonic torsional loads using a very low exciting frequency Ω compared to the first natural torsional frequency ω_{t1} of the member (i.e., $\Omega \approx 0.01\omega_{t1}$), and (*ii*) capable of extracting the eigen-frequencies and eigen-modes from the steady state dynamic response. The present finite element formulation is based on the shape functions which exactly satisfy the homogeneous form of the governing torsional warping coupled equations. This treatment eliminates mesh discretization errors in conventional finite element solutions based on polynomial shape functions and thus converge to the solution using a minimal number of degrees of freedom. As a result, it is observed that, the present nodal results obtained based on the present finite element using a single two-nodded beam element per span yielded results exactly matching those based on the exact closed-form solutions provided by Hjaji and Werfalli [24] up to four significant digits. The numerical results based on the present finite beam element (with two degrees of freedom per node) which accounts for shear deformation due to warping and rotary inertia are compared with exact solutions available in the literature and Abaqus finite beam B13OS element solution which accounts for the effects of shear deformation due to bending. The B31OS beam element (Figure 3) is two-node linear element used for open section members and has seven degrees of freedom per node (i.e., three translations u, v, w, three rotations $\theta_x, \theta_y, \theta_z$ and warping deformation ψ). Moreover, the present finite element formulation is applied to investigate the influence of axial static compressive and tensile forces on the natural torsional frequencies and steady state dynamic of torsional-warping coupled response of open thin-walled doubly symmetric members.

Although excellent nodal degrees of freedom for torsional rotation and warping deformation results are obtained for quasi-static and dynamic responses of the given beam based on one beam element (4 *dof*), but for more general comparison with the Abaqus finite element solution five finite beam elements were used.

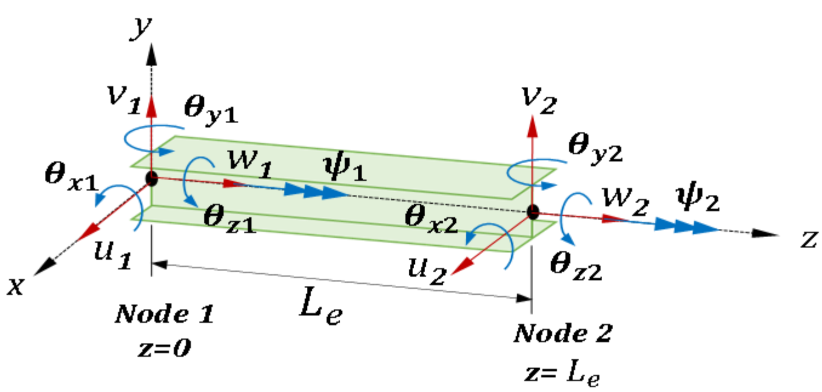


Figure 3: Two-nodded Abaqus B31OS beam element

Example 1- Cantilever I-Beam under Harmonic Torsional Loads

To assess the accuracy and efficiency of the present finite element formulation, a 3.0m cantilever thin-walled beam having doubly-symmetric I-section subjected to various harmonic torsional and warping moments; (i) concentrated end twisting moment

$M_z(L,t) = 1.60e^{i\Omega t} kNm$, concentrated end warping moment $M_w(L,t) = 2.40e^{i\Omega t} kNm^2$ applied at the cantilever free end (i.e., $z = L$), and (ii) uniformly distributed twisting moment $m_z(z,t) = 1.40e^{i\Omega t} kNm/m$ and warping moment $m_w(z,t) = 1.5e^{i\Omega t} kNm^2/m$ applied along the cantilever axis is considered as shown in Figure (4). The geometrical properties of the doubly symmetric cross-section are given in Table (1). For verification purposes, it is required to (a) compute a quasi-static analysis by adopting a very low exciting frequency $\Omega \approx 0.01\omega_{t1}$, and (b) investigate the steady state dynamic torsional-warping coupled response at exciting frequency $\Omega = 1.60\omega_{t1}$, where the first natural torsional frequency of the given cantilever beam is $f_{t1} = 25.60Hz$ $(i.e., \omega_{t1} = 2\pi f_{t1})$.

The numerical results based on the present finite element formulation are compared to the corresponding results based on the exact solution available in the literature [24] and Abaqus finite beam element solution. In Abaqus finite element model, the thin-walled beam is modelled using 100 B31OS elements (707 dof) along the longitudinal axis of the cantilever beam to approach the accuracy of this example. In contrast, the present finite element uses a single beam element (4 *dof*) to capture the exact solution. In this example, the nodal degrees of freedom results obtained from the present finite element formulation use five beam elements (12 *dof*) to exhibit more comparison with Abaqus finite element solution (707 *dof*).

Table 1: Geometric and properties of doubly symmetric thin-walled I-beam

$E = 200GPa$	$A = 7420mm^2$	$I_{xx} = 87.10 \times 10^6 mm^4$	$I_{yy} = 18.82 \times 10^6 mm^4$
$G = 80GPa$	$J = 373.7 \times 10^3 mm^4$	$C_w = 268.0 \times 10^9 mm^6$	$D_{ww} = 77.94 \times 10^6 mm^4$

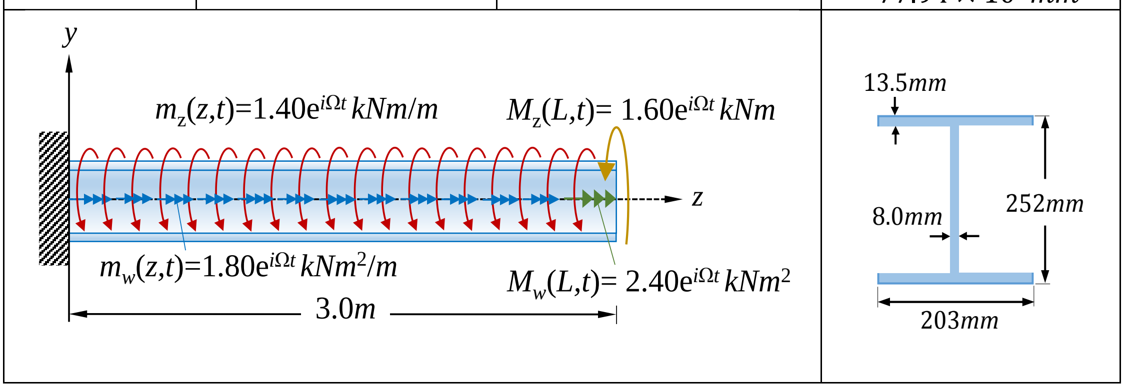


Figure 4: A cantilever thin-walled I-beam under various twisting and warping moments

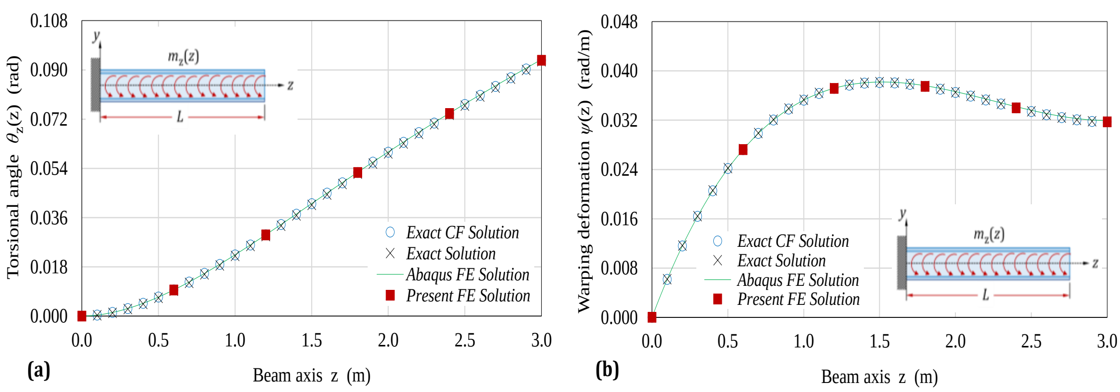
Quasi-Static Response Analysis

To approach the quasi-static response of the cantilever I-beam subjected to various harmonic twisting and warping moments, the exciting frequency is taken significantly lower than the first natural transverse frequency, i.e., $\Omega \approx 0.01\omega_{t1} = 1.608\ rad/sec$. Table (2) provides the quasi-static response results for the torsional angle and warping deformation at the beam free end $(z = L)$. It is obvious that the nodal torsional rotation angle and warping deformation results obtained from the present finite element solution (PS) based on a single beam element are in excellent agreement with exact solution (ES) in [24] and Abaqus beam element model (AS). This is a natural outcome of the fact that the present finite element solution is based on the shape functions which exactly satisfy the homogeneous form of the coupled torsional-warping static equations, which in turn eliminates discretization errors induced in the conventional finite element formulations. The present finite element formulation yields very slightly higher and lower values than those based on exact solution and Abaqus beam model, respectively.

Table 2: Static results for torsional and warping deformations at cantilever free end

Type of load Static response $\Omega \approx 0.001\omega_{t1}$	Function type ($x10^{-3}$)	Exact solution (ES)	Abaqus solution (AS) (707 *dof*)	Present FE solution (PS) (4 *dof*)	Difference (%) (PS-ES)/PS	Difference (%) (PS-AS)/PS
$m_z(z,t) = 1.40e^{i\Omega t} kNm/m$	$\theta_z(L)$	93.47	93.79	93.52	0.05%	-0.29%
	$\psi(L)$	31.69	31.96	31.71	0.06%	-0.79%
$m_w(z,t) = 1.80e^{i\Omega t} kNm^2/m$	$\theta_z(L)$	101.2	101.5	101.3	0.10%	-0.20%
	$\psi(L)$	47.67	47.89	47.70	0.06%	-0.40%
$M_z(L,t) = 1.60e^{i\Omega t} kNm$	$\theta_z(L)$	90.64	90.71	90.68	0.04%	-0.03%
	$\psi(L)$	42.17	42.22	42.20	0.07%	-0.05%
$M_w(L,t) = 2.40e^{i\Omega t} kNm^2$	$\theta_z(L)$	63.26	63.33	63.30	0.06%	-0.05%
	$\psi(L)$	58.72	58.77	58.74	0.03%	-0.05%
$m_z(z,t), m_w(z,t), M_z(L,t), M_w(L,t)$	$\theta_z(L)$	167.6	167.8	167.7	0.06%	-0.06%
	$\psi(L)$	95.91	96.30	95.95	0.04%	-0.36%

Although excellent nodal torsional rotation and warping deformation results are obtained for quasi-static response of the given cantilever using one beam element (4 *dof*), but for more general comparison with the Abaqus finite element solution (707 *dof*) five finite elements are used. The nodal torsional rotation $\bar{\Theta}_{zn}$ and warping deformation $\bar{\Psi}_n$ (where $n = 1,2,3,4,5$) are shown in Figures (5a,c,e,g,i) and (5b,d,f,h,j), respectively, for cantilever beam under various harmonic twisting and warping moments. Four solutions, based on the exact closed-form solution [24], exact solution [39], Abaqus finite beam B31OS element, and the present finite element solution are overlaid on the same diagrams for comparison. It is noted that, the present finite element formulation provides excellent agreement with other three solutions. As a general observation, the present finite element solution is successful at capturing the static torsional-warping coupled response of the structural thin-walled member.



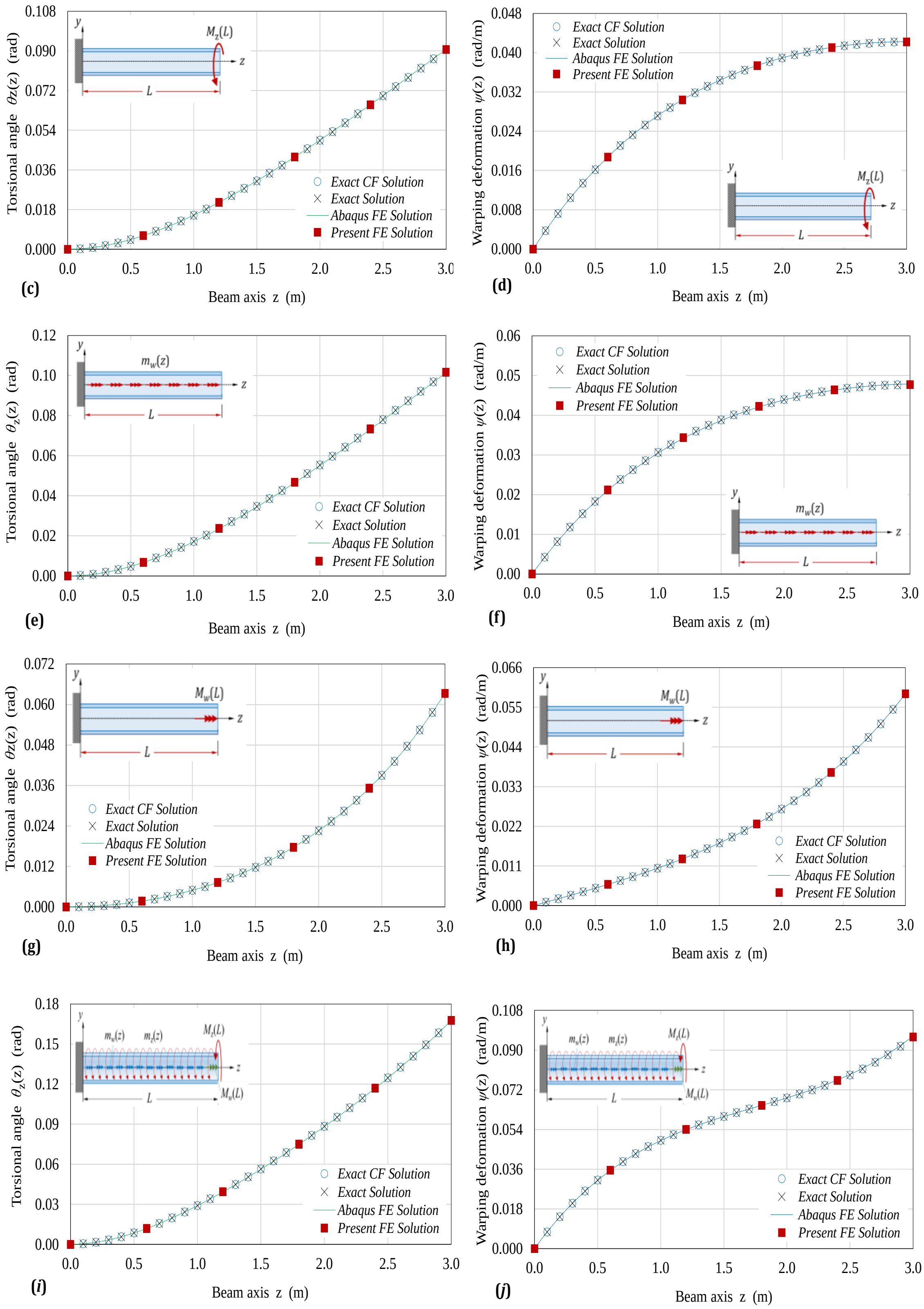


Figure 5: Quasi-static of torsional-warping coupled response for cantilever thin-walled I-beam under various torsional moments

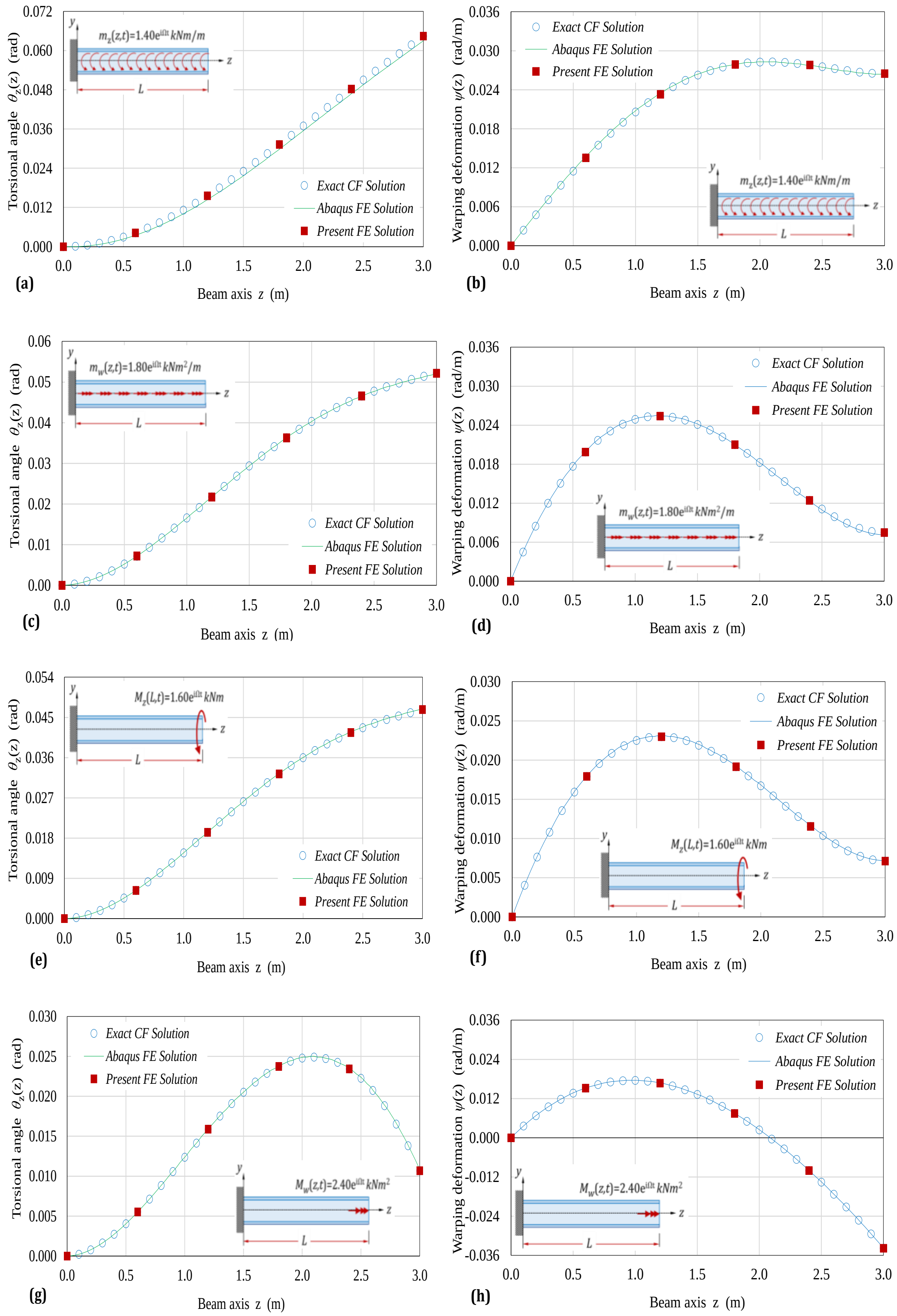
Dynamic Response Analysis

The steady state response dynamic analysis for nodal torsional rotation $\bar{\Theta}_{zn}$ and warping deformation $\bar{\Psi}_n$ of cantilever open thin-walled I-beams subjected to various harmonic twisting and warping moments captured by using exciting frequency $\Omega = 1.60\omega_{t1} = 257.4\ rad/sec$ are provided in Table (3). The nodal degrees of freedom results at cantilever free end obtained using three different solutions: (*i*) the present finite element formulation (PS) using a single beam element with 4 *dof*, (*ii*) the exact closed-form solution (ES) in [24], and (*iii*) Abaqus beam element model (AS) using one hundred B31OS element with 707 *dof* in order to achieve the solution accuracy. It is observed that, the nodal results obtained from the present finite element are found exactly identical to the exact closed-form solution. It is also seen that the present finite element formulation (which captures the shear deformation due to warping torsion) predict results in close agreement with the results obtained from Abaqus beam B31OS element solution (which captures only the shear deformation due to bending). In other words, the results obtained from the present finite element solution using one beam element (4 *dof*) are differed from -0.03% to -3.72% from those based on Abaqus finite beam solution using one hundred B31OS element.

Table 3: Dynamic results for nodal torsional and warping functions at cantilever end

Type of load	Function type (x10^{-3})	Exact solution (ES)	Abaqus solution (AS)	Present FE solution (PS)	Difference (%)	Difference (%)
Dynamic response $\Omega = 1.60\omega_{t1}$					(FE-ES)/FE	$\lvert(\mathrm{FE}-\mathrm{AS})/\mathrm{FE}\rvert$
$m_z(z,t) = 1.40e^{i\Omega t}kNm/m$	$\theta_z(L)$	64.42	64.60	64.42	0.00%	-0.28%
	$\psi(L)$	26.48	26.40	26.48	0.00%	0.30%
$m_w(z,t) = 1.80e^{i\Omega t}kNm^2/m$	$\theta_z(L)$	52.18	52.00	52.18	0.00%	0.34%
	$\psi(L)$	7.475	7.230	7.475	0.00%	3.28%
$M_z(L,t) = 1.60e^{i\Omega t}kNm$	$\theta_z(L)$	46.75	46.90	46.75	0.00%	-0.32%
	$\psi(L)$	7.121	7.180	7.121	0.00%	-0.83%
$M_w(L,t) = 2.40e^{i\Omega t}kNm^2$	$\theta_z(L)$	10.68	10.80	10.68	0.00%	-1.12%
	$\psi(L)$	-33.79	-33.80	-33.79	0.00%	-0.03%
$m_z(z,t), m_w(z,t), M_z(L,t), M_w(L,t)$	$\theta_z(L)$	80.53	80.40	80.53	0.00%	0.16%
	$\psi(L)$	-6.962	-7.220	-6.961	-0.01%	-3.72%

For more comparison, the steady state dynamic responses represented the nodal torsional rotation $\bar{\Theta}_{zn}$ and warping deformation $\bar{\Psi}_n$ (for $n = 1,2,3,\ldots,12$) for cantilever thin-walled I-beam under various harmonic torsional and warping moments with exciting frequency $\Omega = 1.60\omega_{t1} = 257.4\ rad/sec$ are displayed against the beam coordinate z as illustrated in Figures (6a,c,e,g,i) and (6b,d,f,h,j), respectively. The nodal degrees of freedom results based on three solutions: (*i*) the finite element developed in the present study (PS), (*ii*) exact closed-form solution (ES) in [24], and (*iii*) Abaqus beam element (AS) using 100 B31OS elements, are plotted on the same diagrams for the sake of comparison. It is noted that, the present finite element formulation (using five beam elements with 12 *dof*) provides an excellent agreement with that based on Abaqus beam solution (using one hundred B31OS beam element with 707 *dof*) at a fraction of the computational and modelling cost. Again, this is the natural outcome that the present beam element is based on the shape functions which exactly satisfy the exact solution of the coupled field equations, which in turn eliminates discretization errors encountered under other interpolation schemes.



Torsional angle θz(z) (rad)
Warping deformation ψ(z) (rad/m)
Beam axis z (m)
Exact CF Solution
Abaqus FE Solution
Present FE Solution
mz(z,t)=1.40e^iΩt kNm/m
mw(z,t)=1.80e^iΩt kNm²/m
Mz(L,t)=1.60e^iΩt kNm
Mw(z,t)=2.40e^iΩt kNm²
y
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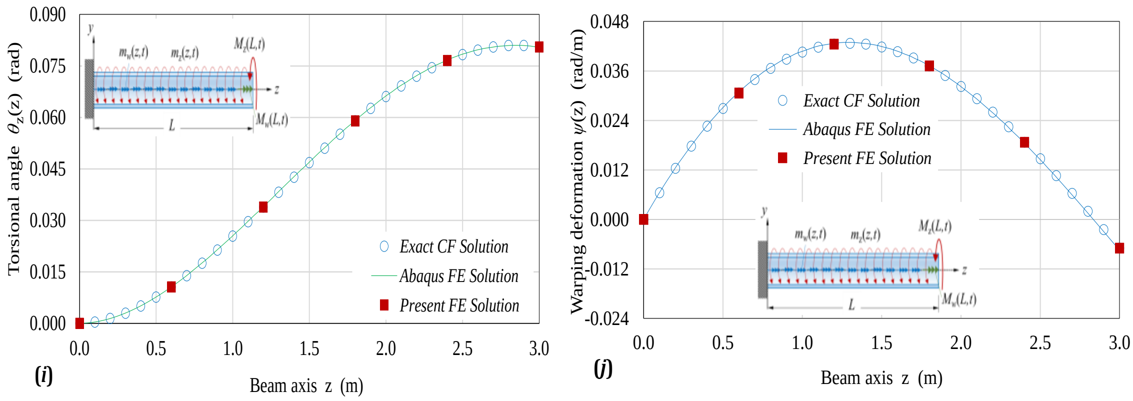


Figure (6): Steady state dynamic torsional-warping coupled responses for cantilever thin-walled I-beam under various harmonic twisting and warping moments

Example (2): Effect of Axial Static Force

In order to investigate the effect of axial static forces on the quasi-static, dynamic analyses and natural torsional frequencies for the coupled torsional-warping responses, a 5000mm simply supported open thin-walled I-beam with fork-type end supports subjected to harmonic distributed twisting moment $m_z(z,t) = 1.50e^{i\Omega t} kNm/m$ and warping moment $m_w(z,t) = 1.80e^{i\Omega t} kNm^2/m$ and axial static force P_{zo} is considered as shown in Figure (7). The simply supported beam is unrestrained along its length while the fork supports prevent the cross-section from torsional rotation and moving laterally but allow for the warping. The material of the beam is steel with $E = 200GPa$, $G = 78GPa$, and material density $\rho = 7800kg/m^3$, while the geometrical properties of the cross-section are given in Table (4).

This example is provided to: (1) compute the quasi-static response analysis by adopting an exciting frequency $\Omega \approx 0.001\omega_{t1} = 0.1789\ rad/sec$, (2) conduct a steady state dynamic analysis to extract the natural torsional frequencies, (3) conduct a steady state responses for various exciting frequencies ($\Omega = 1.5\omega_{t1}$, $3.5\omega_{t1}$ and $5.5\omega_{t1}$), and (4) investigate the effect of axial static force on natural torsional frequencies, quasi-static and steady state dynamic responses, where the first natural torsional frequency ω_{t1} of the given I-beam is $\omega_{t1} = 178.9\ rad/sec$.

Table 4: Geometric and properties of doubly symmetric thin-walled I-section beam

$A = 4560mm^2$	$I_{xx} = 24.27 \times 10^6 mm^4$	$I_{yy} = 3.456 \times 10^6 mm^4$
$J = 19.42 \times 10^3 mm^4$	$C_w = 24.39 \times 10^9 mm^6$	$D_{ww} = 17.52 \times 10^6 mm^4$

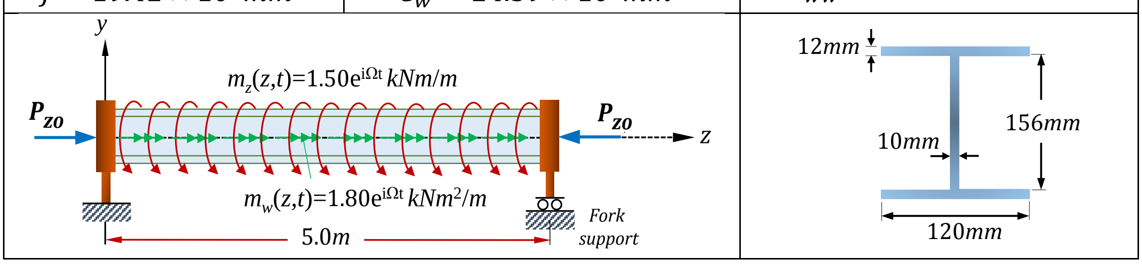


Figure 7: A Fork-supported I-beam under harmonic distributed twisting and warping moments

For the sake of validation, the numerical nodal torsional rotation and warping deformation results obtained from the finite element solution using five beam elements (with 12 *dof*) developed in this study are compared with the Abaqus finite element model and exact closed-form solution [24]. The fork-supported beam is modelled in Abaqus solution by using 160 B31OS beam elements (i.e., a total of 1127 *dof*) along the beam axis to achieve the required accuracy.

Quasi-Static Response Analysis

The quasi-static analysis for torsional-warping coupled response of simply-supported I-beam subjected to harmonic twisting moment $m_z(z,t) = 1.50e^{i\Omega t} kNm/m$ and warping moment $m_w(z,t) = 1.80e^{i\Omega t} kNm^2/m$ is captured by using very low exciting frequency $\Omega \approx 0.001\omega_1 = 0.1789\, rad/sec$ related to the first natural torsional frequency ω_{t1} of the given I-beam (i.e., $\omega_{t1} = 178.9\, rad/sec$). The nodal static results for coupled torsional-warping response are computed using three different solutions: (a) the exact closed-form solution presented in [24], (b) the finite element solution using five beam elements, and (c) Abaqus finite element model using 160 beam B31OS elements. Even though, the present finite element formulation based on two beam element (4 *dof*) provided excellent results but for the sake of comparison five beam elements with 12 *dof* were used.

The nodal torsional rotation $\bar{\Theta}_{zn}$ and warping deformation function $\bar{\Psi}_n$ (where $n = 1,2,3, \ldots ,12$) as illustrated in Figure (8), based on present finite element solution, Abaqus beam B31OS model, and exact solution [24], are overlaid on the same diagrams for comparison. As a general remark, Figure (8) shows excellent agreement between all three solutions. Furthermore, the developed finite element results based on five beam elements (12 *dof*) shows excellent agreement with those based on the Abaqus finite model solution using 160 beam B31OS elements (1127 *dof*). Again, the present finite element solution is successful at capturing the static response of the given beam.

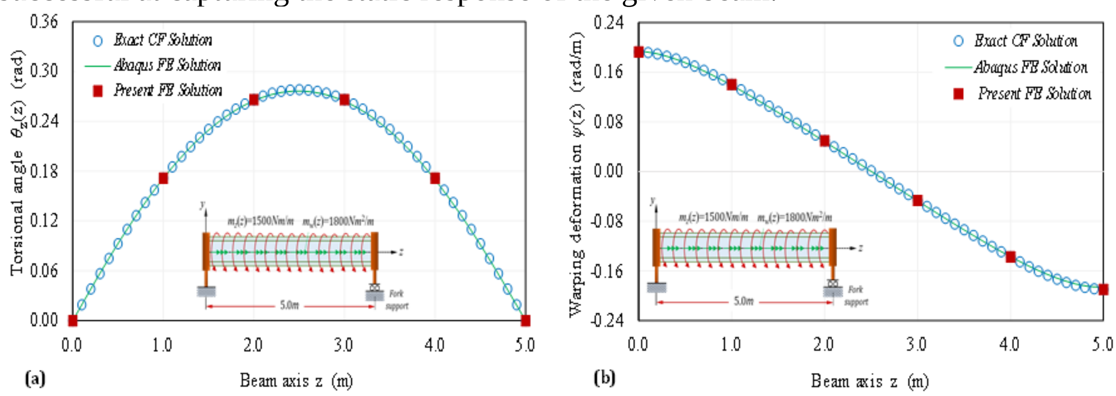


Figure 8: Static analysis for torsional-warping coupled response of fork-supported beam under distributed harmonic twisting and warping moments with $\Omega \approx 0.001\omega_{t1}$

Dynamic Response Analysis

The steady state torsional-warping responses for simply-supported I-beam subjected to distributed harmonic twisting and warping moments having three different values of exciting frequencies ($\Omega_1 = 1.5\omega_{t1} = 268.4\, rad/sec$, $\Omega_2 = 3.5\omega_{t1} = 626.2 rad/sec$, and $\Omega_3 = 5.5\omega_{t1} = 984.0 rad/sec$) are illustrated in Figures (9a,c,e) and (9b,d,f), respectively. The nodal torsional rotation and warping deformation results based on the present formulation are compared with those based on Abaqus beam model and exact solutions. It is observed that results obtained from the finite element formulation developed using five beam elements with 12 *dof* provide excellent agreement with Abaqus beam model using 160 B31OS elements (1127 *dof*).

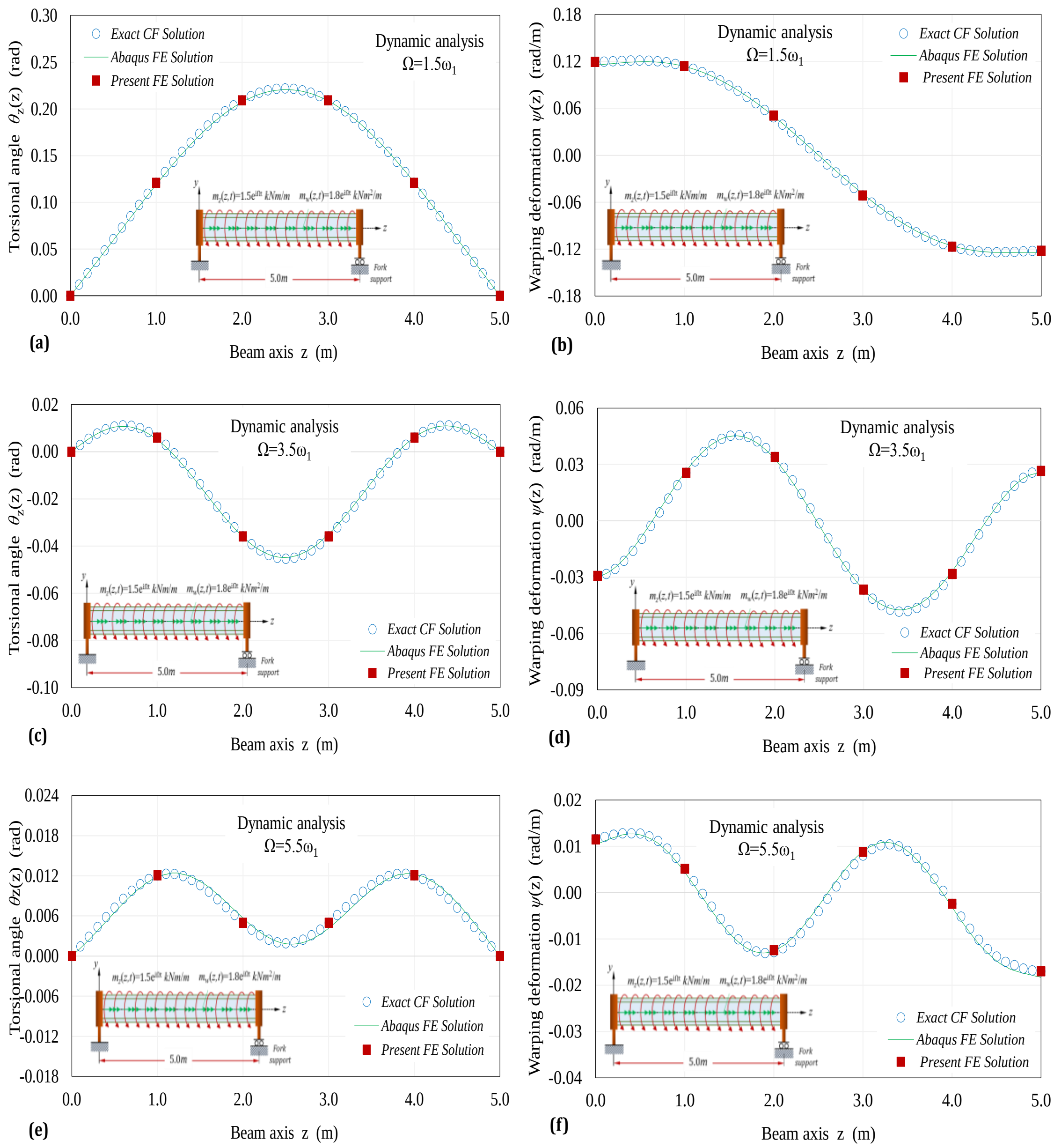


Figure 9: Dynamic analyses for torsional-warping coupled responses of fork-supported I-beam under harmonic twisting and warping moments with various exciting frequencies

Steady State Dynamic Analysis - Natural Torsional Frequencies

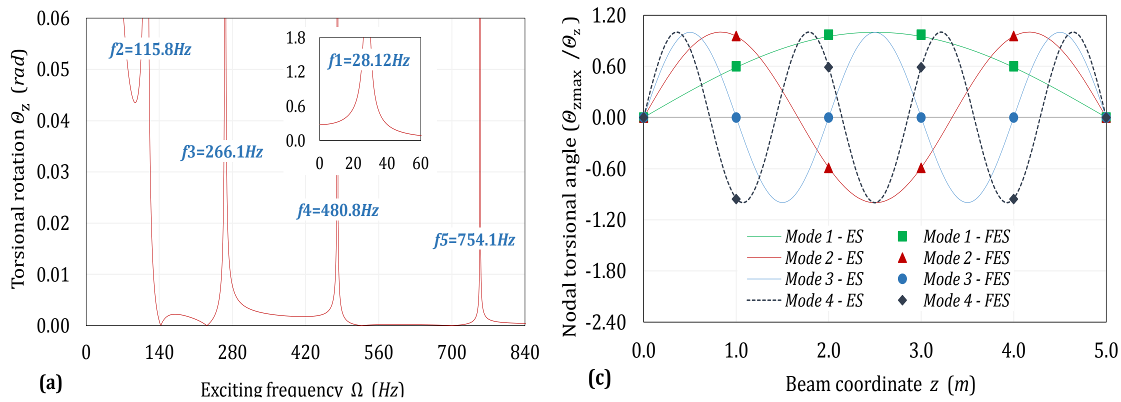
Under distributed harmonic twisting moment $m_z(z,t) = 1.50e^{i\Omega t} kNm/m$ and warping moment $m_w(z,t) = 1.80e^{i\Omega t} kNm^2/m$, the natural frequencies related to coupled torsional-warping response are extracted from the steady state torsional response analyses in which the exciting frequency f_t varying from nearly zero to $840 Hz$. Figure (10a-b) show the nodal torsional rotation $\bar{\Theta}_{z2}$ and warping deformation $\bar{\Psi}_2$ at node 2 against the exciting frequency. The natural torsional frequencies are then obtained at the peaks of the torsional rotation-frequency relationship. Peaks on both diagrams (Fig. 10a and 10b) indicate resonance and are thus indicators of the natural torsional frequencies of the beam. Each peak indicates resonance and thus identifies natural torsional frequencies of the given simply supported beam. Then, the first five natural torsional-warping frequencies extracted from the peaks are provided in Table (5). Table (5) presents the first five natural frequencies extracted from the steady state torsional-warping dynamic responses obtained based on three different solutions: the exact solution, the present finite element formulation (both solutions capture the shear deformation effect due to warping) and Abaqus beam model (which capture only the shear deformation due to bending).

As can be indicated form Table (5), the natural torsional frequencies predicted by Abaqus beam model slightly differ from those based on the present finite element and exact solutions by 0.12%-1.35%. Moreover, the solution predicted by Abaqus B31OS beam model showed slightly lower natural torsional frequencies than other solutions and this is since the shear deformation due to warping is not captured by such model.

Table 5: First four natural torsional frequencies of simply supported thin-walled I-beam under distributed harmonic twisting moment

Frequency No.	Natural torsional frequencies in *Hz*				
	Present finite element (PS)	Exact solution (ES)	Abaqus solution (AS)	% Difference = [PS-ES]/PS	% Difference = [PS-AS]/PS
1	28.12	28.12	27.74	0.0%	1.35%
2	115.8	115.8	114.5	0.0%	1.12%
3	266.1	266.1	264.8	0.0%	0.49%
4	480.8	480.8	480.2	0.0%	0.12%

The first four steady state torsional-warping mode shapes of the simply supported I-beam under the given harmonic torsional and warping moments are illustrated in Figure (10c-d). For comparison, the normalized steady state torsional-warping modes $(\bar{\Theta}_{zn}/\bar{\Theta}_{znmax})$ and $(\bar{\Psi}_n/\bar{\Psi}_{nmax})$ based on the present finite element formulation and exact closed-form solution [24] are plotted on the same diagrams for the first four torsional exciting frequencies: $f_{t1} = 28.12Hz,\ f_{t2} = 115.8Hz, f_{t3} = 266.1Hz, and\ f_{t4} = 480.8Hz$. Nodal results for torsional rotation $\bar{\Theta}_{zn}$ and warping deformation $\bar{\Psi}_n$ obtained from the present formulation exhibit excellent agreement.



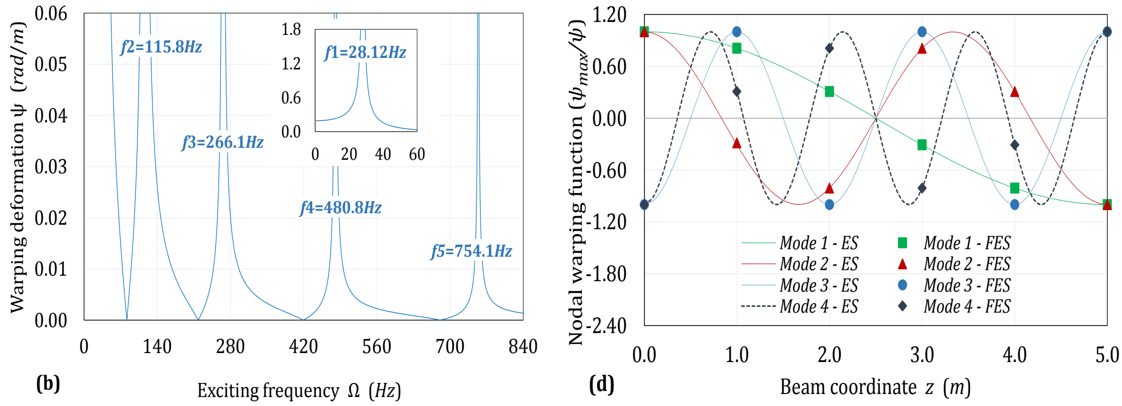


Figure 10: Natural torsional frequencies and mode shapes of fork-supported I-beam under distributed harmonic twisting moment

Axial Static Force Effects on Natural Torsional Frequencies

The simply supported thin-walled I-beam under uniformly harmonic distributed twisting and warping moments is subjected to axial static force P_{zo} is considered to investigate the influence of axial static tensile and compressive forces on the natural torsional frequencies. The axial static force is acted through the centroid of the cross-section. The first five natural torsional frequencies extracted from the steady state dynamic response analyses of the given beam are plotted in Figure (11) for different values of axial forces$(i.e., P_{zo} = -2.0MN, -1.0MN, 0.0MN, +1.0MN, +2.0MN)$. It is observed that, the results natural torsional frequencies given in Figure (11) show an excellent agreement between the predictions of natural torsional-warping frequencies based on the present finite element solution (*FES*) and exact solution (*ES*). It is also seen that the natural torsional-warping frequencies increases with the increase of axial static tensile forces, while an increase of axial compressive static force leads to decrease the natural torsional frequencies. In addition, it is observed that as the order of the natural frequency increases, the effect of axial static force on the torsional natural frequencies becomes more pronounced. Thus, the effect of axial static force on the high order natural torsional frequencies is more significant than the lower natural frequencies. This leads to conclude that, the current results for axial static force effects on natural torsional frequencies give the same concluding remarks were obtained in the previous study [24].

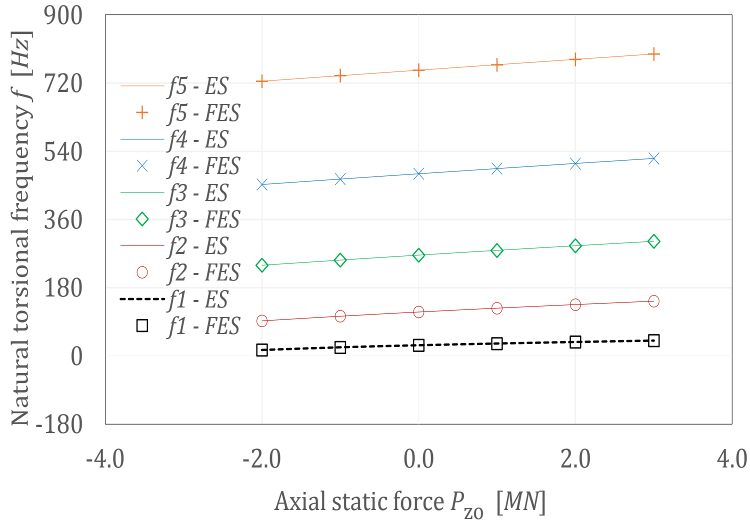


Figure 11: Axial static force effect on the natural torsional frequencies of simply-supported I-beam

Axial Static Force Effects on Quasi-Static Response

The axial static force influence on the quasi-static and dynamic torsional-warping coupled response of the simply-supported I-beam is investigated in Figure (12a-b) by using very low exciting frequency $\Omega \approx 0.001\omega_{t1} = 0.1789 rad/sec$ for quasi-static and exciting frequency $\Omega = 1.80\omega_{t1} = 322.0 rad/sec$ for dynamic analysis. The diagrams in Figure (12) are plotted for different values of axial static force that changed from compression to tension $(i.e., P_{zo} = -2.0MN, -1.0MN, 0.0MN, +1.0MN, +2.0MN)$. It is noted that, as the values of applied axial force increased, the static torsional rotation and warping deformation responses are decreased. Additionally, it is obvious that the axial tensile force has a stiffening effect while the compressive force has a softening effect on the coupled torsional-warping static response. Therefore, the axial static compressive force has more significant influence on the quasi-static torsional-warping responses for the simply supported I-beam than that of the corresponding axial static tensile force. In addition, the results obtained using the present finite element solution (*FES*) are in excellent agreement with the results of Abaqus finite element solution (*AFE*).

Axial Static Force Effects on Dynamic Response

The steady state dynamic response nodal results for torsional rotation $\bar{\Theta}_{zn}$ and warping deformation $\bar{\Psi}_n$ versus the beam coordinate axis z are illustrated in Figure (12c-d) for exciting frequencies $\Omega = 322.0 rad/sec$, respectively. The effect of axial static force P_{zo} on the steady state torsional-warping dynamic responses of the simply supported I-beam is investigated as shown. Again, the nodal results based on the present finite element solution (*FES*) using five beam elements (12 *dof*) are in excellent agreement with Abaqus finite element beam solution (*AFE*) using 160 B31OS beam element (1127 *dof*). The amplitudes of the nodal torsional displacement $\bar{\Theta}_{zn}$ and warping deformation $\bar{\Psi}_n$ decrease as the axial static force changes from tension $P_{zo} = 2.0MN$ to compression $P_{zo} = -2.0MN$. In other words, the results indicated that the axial static compressive force softens the beam whereas the tensile force stiffens the beam. This observation exhibits that the axial static force has an opposite effect to that of the quasi-static torsional-warping response.

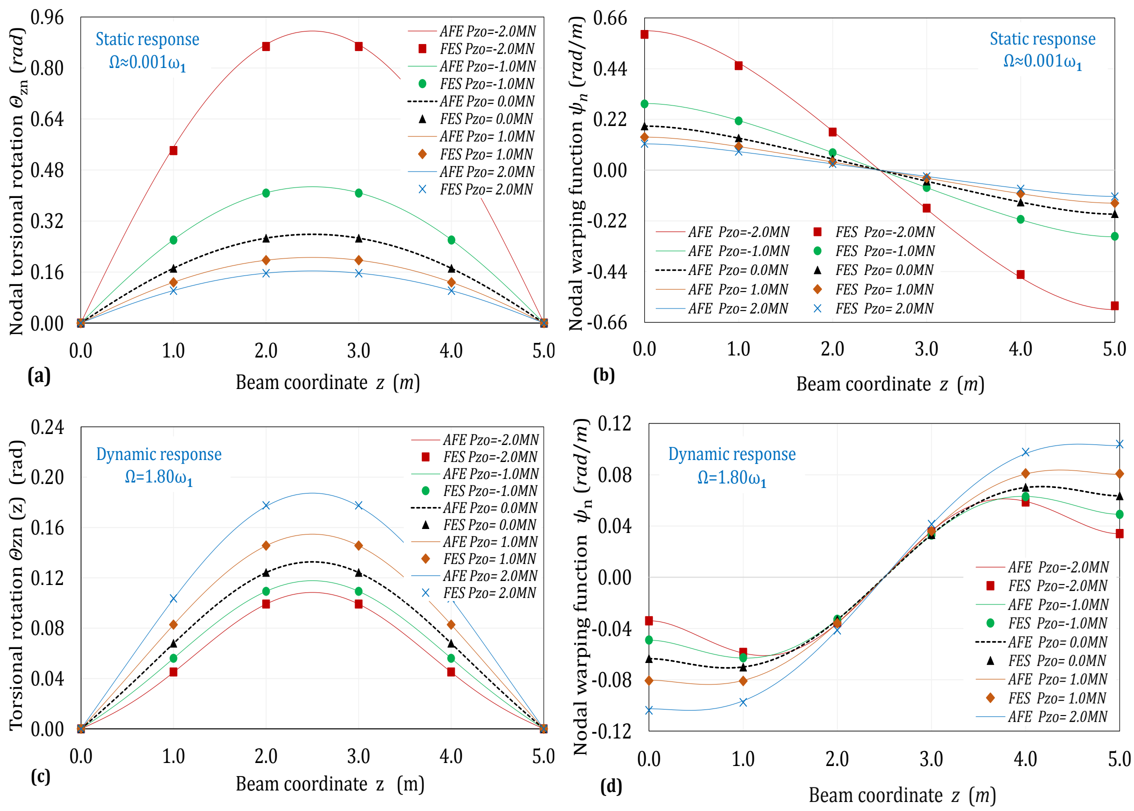


Figure 12: Axial static force effects on static and dynamic torsional-warping analyses of simply supported I-beam under harmonic twisting and warping moments

Example (3) – Validation of Finite Element Formulation

This example is aimed at establishing the capability of the present finite element developed in this study to predict the nodal torsional and warping functions for coupled torsional-warping quasi-static and dynamic responses. A 5.0m fixed-fork open thin-walled I-beam is subjected to various torsional and warping harmonic moments; distributed twisting moment $m_z(z,t) = 0.50e^{i\Omega t} kNm/m$ and distributed warping moment $m_w(z,t) = 0.80e^{i\Omega t} kNm^2/m$ acting along beam axis, while the concentrated twisting moments $M_{z1}(1.25m,t) = 1.0e^{i\Omega t} kNm$ and $M_{z2}(3.75m,t) = 2.5e^{i\Omega t} kNm$ applied as shown in Figure (13). The geometric properties of the beam section are provided in Table (6). It is required to assess the accuracy and efficiency of the present finite element formulation solution in determining the nodal degrees of freedom for quasi-static response $(\Omega \approx 0.001\omega_{1t})$ and steady state dynamic responses with various exciting frequencies $(\Omega = 50, 100, 150 \; and \; 200Hz)$ (under an exciting frequency, where the first natural torsional frequency of the given beam is $f_{1t} = 28.49Hz$. $E = 210GPa$, $\rho = 7800kg/m^3$

Table 6: Geometric and properties of doubly symmetric thin-walled I-section beam

$A = 6500mm^2$	$I_{xx} = 45.25 \times 10^6 mm^4$	$I_{yy} = 10.25 \times 10^6 mm^4$
$J = 421.7 \times 10^3 mm^4$	$C_w = 87.62 \times 10^9 mm^6$	$D_{ww} = 41.07 \times 10^6 mm^4$

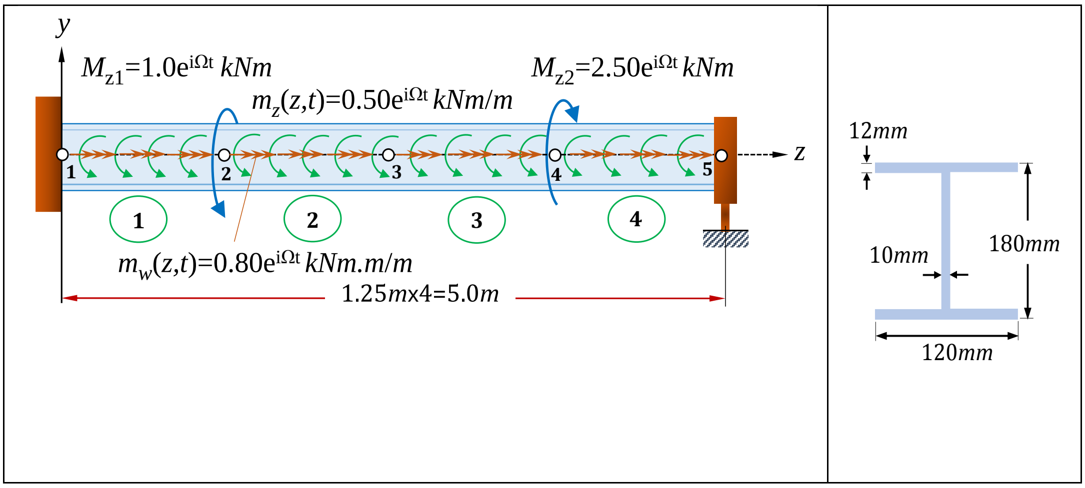


Figure 13: A fixed-fork thin-walled I-beam under various harmonic twisting and warping harmonic moments

Two solutions are provided for this problem to perform the quasi-static and dynamic analyses. The first solution is based on the Abaqus finite beam element of 160 beam B31OS elements in which a total of 1,127 degrees of freedom were needed to eliminate the mesh discretization errors and achieve the required accuracy of the solution. The second solution is based on the present finite element formulation, in which the beam is subdivided into only four beam elements along the beam coordinate, i.e., the present finite element model has only 10 degrees of freedom.

Quasi-Static Torsional-Warping Response Analysis

The nodal torsional rotation θ_{zi} and warping deformation ψ_i (for $i=1,2,3,....,10$) are provided in Figures (14a) and (14b), respectively, for the torsional-warping coupled static response of the given beam based on three solutions; Abaqus beam model solution, finite element solution based on exact shape functions [39], and present finite element formulation. It is observed form these figures that, the nodal torsional rotation and warping deformation functions predicted by the present finite element model using four beam elements provide an excellent agreement with those based on Abaqus finite solution using 160 beam B31OS elements and static solution using four finite beam elements.

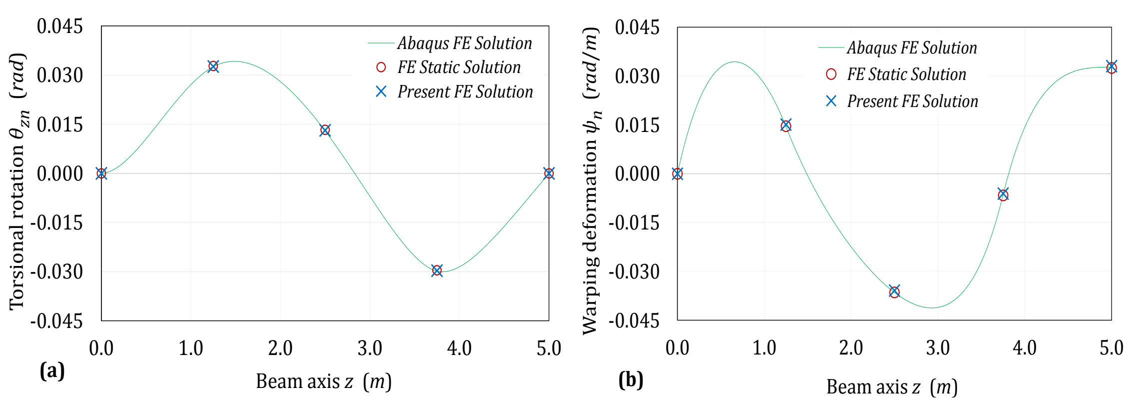
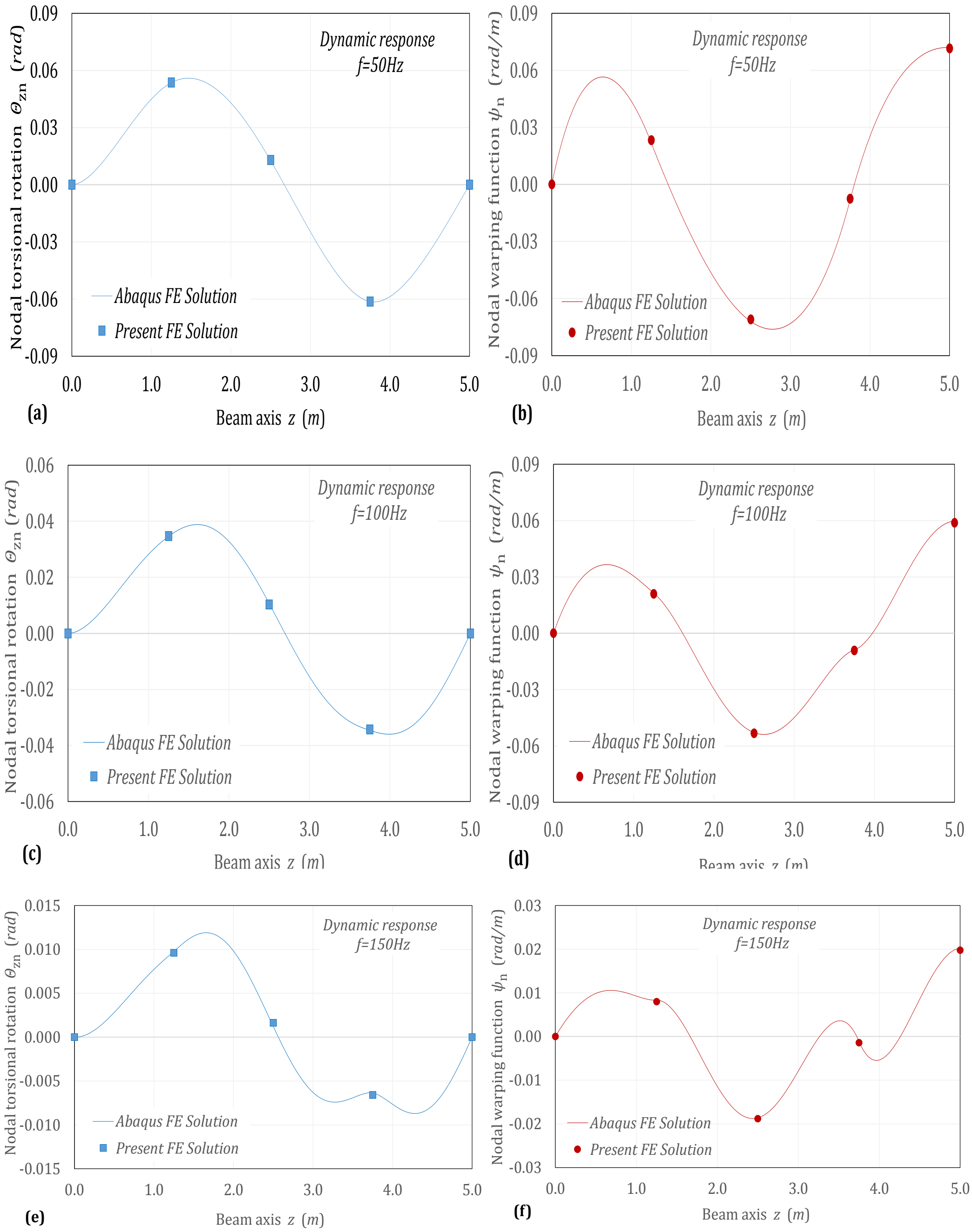


Figure 14: Static torsional-warping coupled analysis of fixed-fork I-beam under various harmonic twisting and warping moments

Dynamic Torsional-Warping Response Analysis

The steady state dynamic torsional-warping responses for the given beam subjected to various distributed harmonic twisting and warping moments having four different values of exciting frequencies ($f_1 = 50Hz$, $f_2 = 100Hz$, $f_3 = 150Hz$, and $f_4 = 200Hz$) are provided and illustrated in Figures (15a-h), respectively.



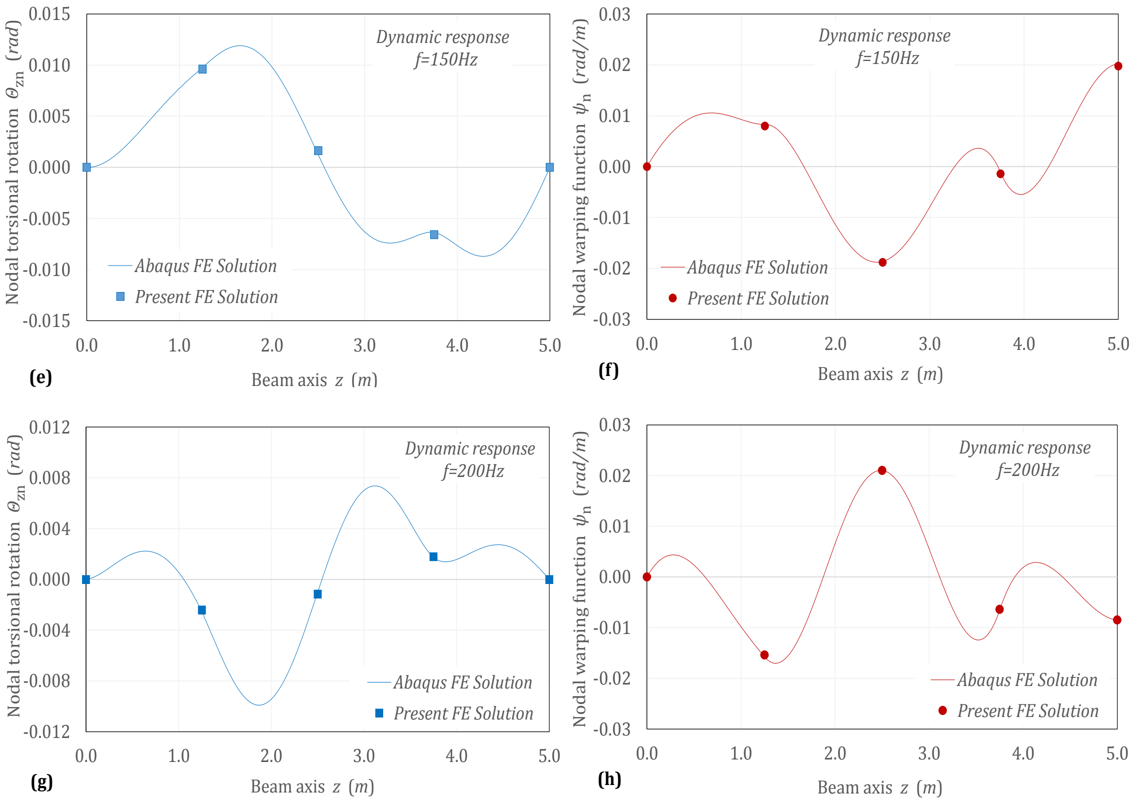


Figure 15: Torsional-warping coupled dynamic analysis of fixed-fork beam under various twisting and warping moments having different exciting frequencies

The nodal torsional rotation θ_{zi} and warping deformation ψ_i (for $i=1,2,3,...,10$) results based on the present formulation are compared with those based on Abaqus beam model. It is observed that results obtained from the finite element formulation developed using four beam elements with 10 *dof* provide excellent agreement with Abaqus finite beam model using 160 B31OS elements (1,127 *dof*). The computational efforts in the present finite element quasi-static and dynamic solutions are several orders of magnitudes less than that of Abaqus beam model solution. This is a natural outcome of the fact that the present finite element is based on the shape functions, which exactly satisfy the homogeneous form of the governing torsional-warping dynamic equations, which in turn eliminates discretization errors encountered in finite-element formulations.

SUMMARY AND CONCLUSION

1. A super-convergent finite element formulation was developed for open thin-walled beams with doubly symmetric cross-sections under various harmonic torsional and warping moments.
2. The present formulation captures the effects of shear deformation due to non-uniform torsion, warping deformation and rotary inertial effects.
3. The new two-noded beam element is based on shape functions which exactly satisfy the homogeneous solution of the coupled torsional-warping dynamic equations derived in previous study [24].

4. The beam element involves no discretization errors encountered under other interpolation schemes and generally provides excellent results with a significantly smaller number of degrees of freedom.
5. The finite element formulation can efficiently capture the quasi-static and steady state response of open thin-walled beams under various harmonic torsional and warping moments. It is also capable of extracting the eigen-frequencies and eigenmodes from the steady state dynamic response of the structural beam.
6. The finite element formulation developed in the present study offers excellent agreement with ABAQUS finite B31OS beam element at a fraction of the computational and modelling effort.
7. Results demonstrate that the effects of axial static force are more significant on the higher natural torsional-warping frequencies than lower natural frequencies.
8. The axial tensile force has a stiffening effect while the compressive force has a softening effect on the coupled torsional-warping static response, while this observation has opposite influence for the case of dynamic response.

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